Confidentiality Protection and Utility for Contingency Table Data

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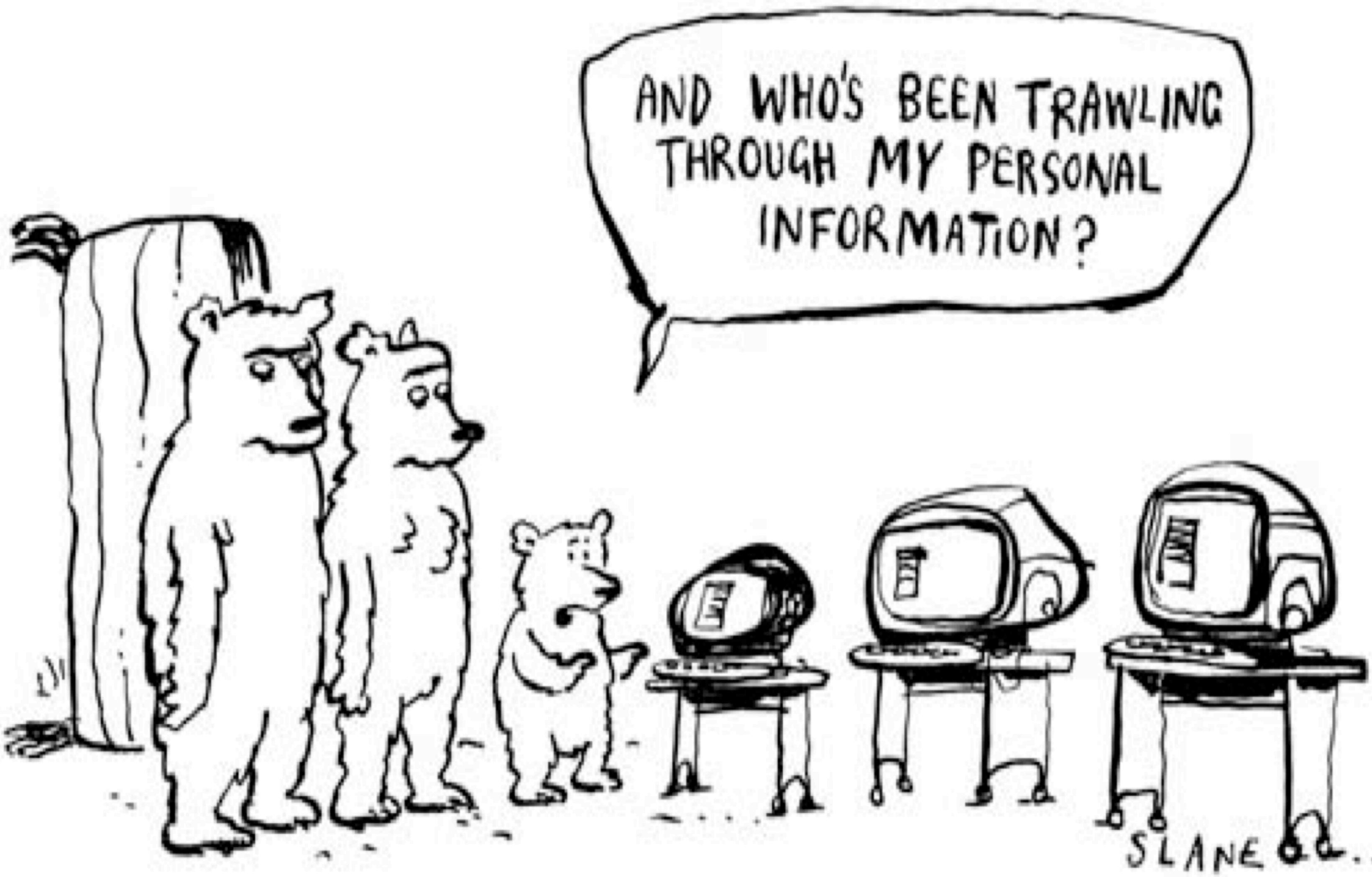
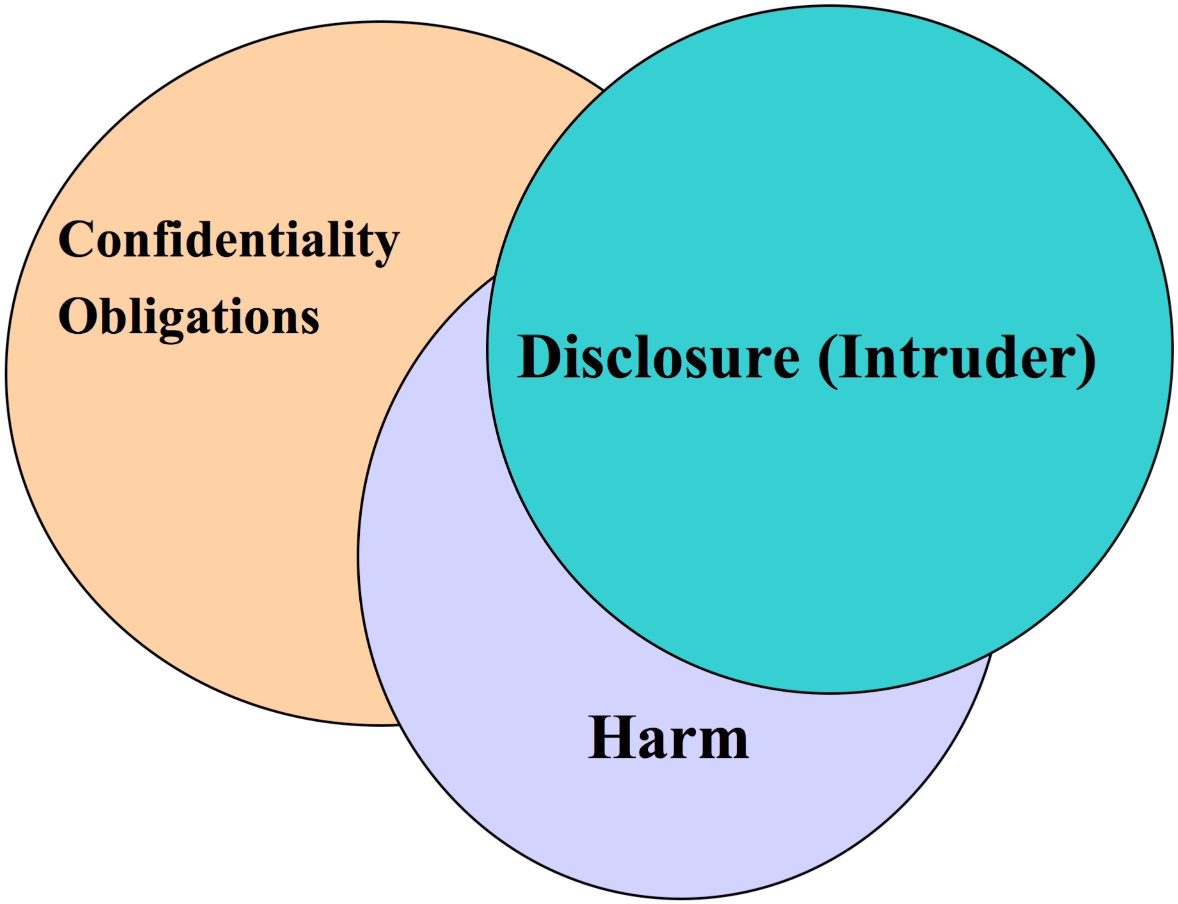
(Joint work with A. Dobra, A. Rinaldo, and Y. Zhou)

Outline

- Privacy and confidentiality
  - Focus individual data (not establishment data)
- Three examples and two problems:
  1. Bounds for cell counts in contingency tables given marginals.
  2. Maximum likelihood estimation for log-linear models.
  - How are they interrelated?
  - What are the mathematical tools? (No details!)
  - Scaling up computations for large sparse tables.

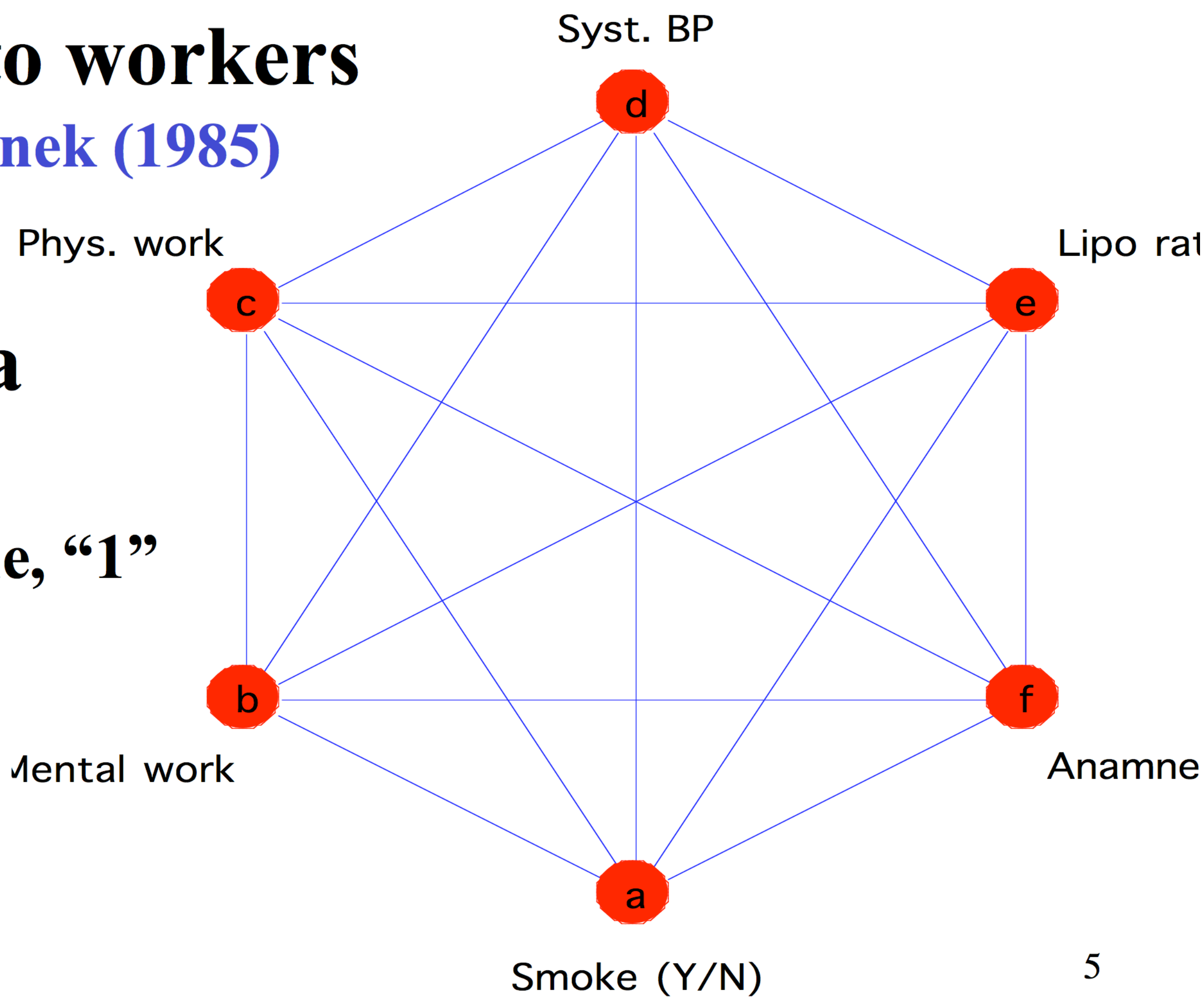

AND WHO'S BEEN TRAWLING THROUGH MY PERSONAL INFORMATION?
SLANE

Issues and Linkages



Ex. 1: Risk Factors for Coronary Heart Disease

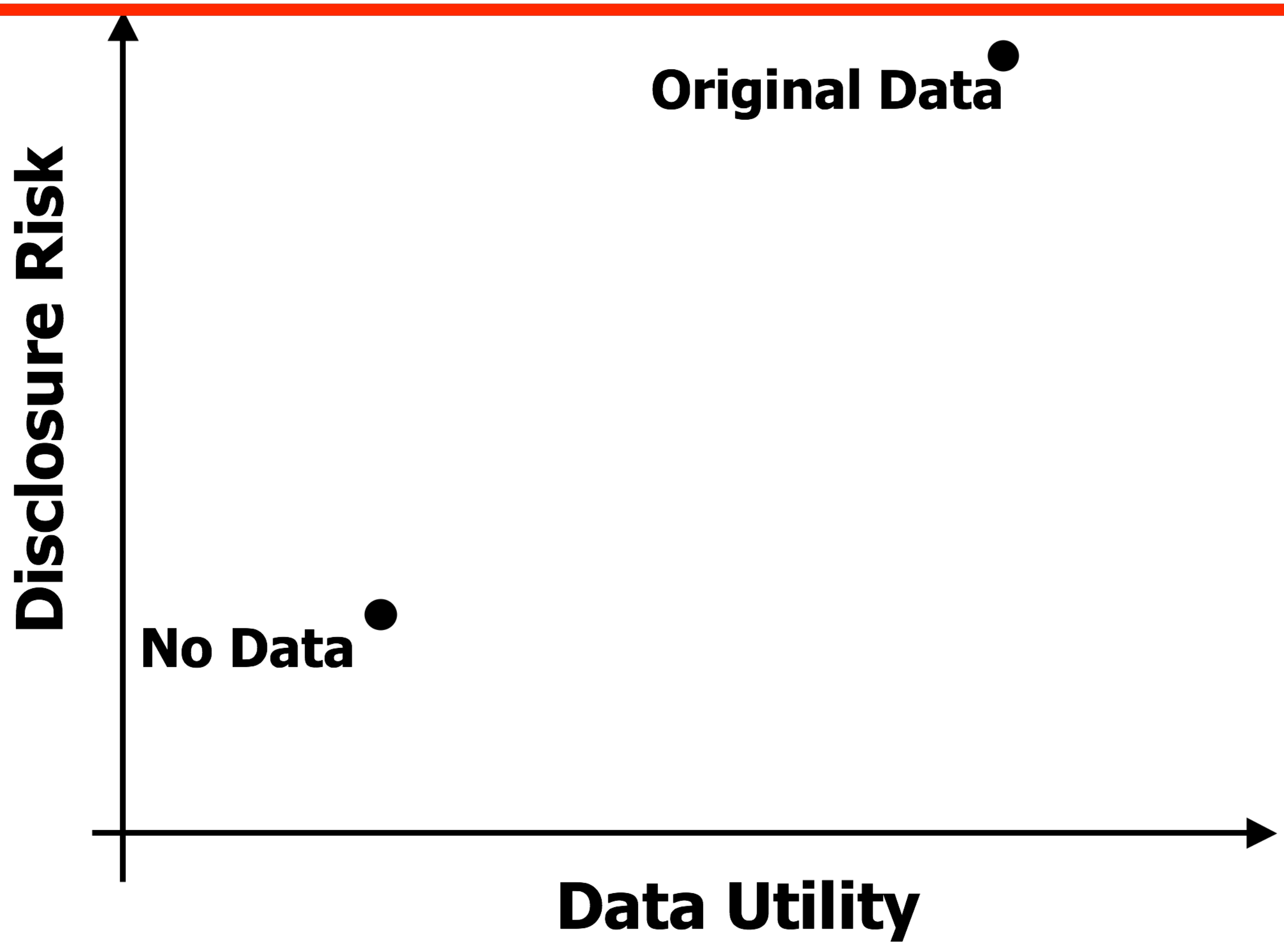
- 1841 Czech auto workers
  Edwards and Havanek (1985)
- 2^6 table
- population data
  - "0" cell
  - population unique, "1"
  - 2 cells with "2"



Ex. 1: The Data

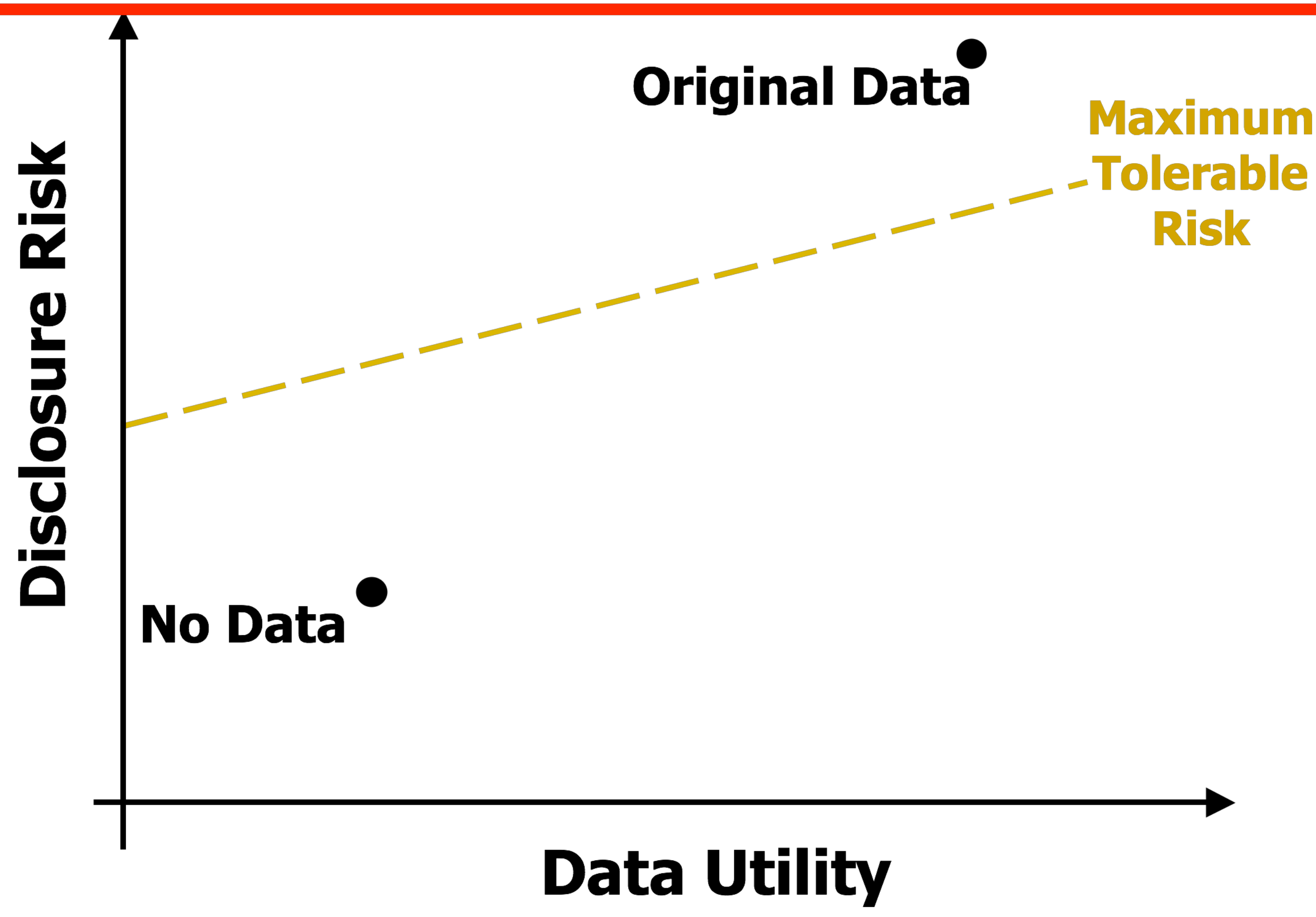
F	E	D	C	B	no		yes	
				A	no	yes	no	yes
neg	< 3	< 140	no		44	40	112	67
			yes		129	145	12	23
		≥ 140	no		35	12	80	33
			yes		109	67	7	9
	≥ 3	< 140	no		23	32	70	66
			yes		50	80	7	13
		≥ 140	no		24	25	73	57
			yes		51	63	7	16
pos	< 3	< 140	no		5	7	21	9
			yes		9	17	1	4
		≥ 140	no		4	3	11	8
			yes		14	17	5	2
	≥ 3	< 140	no		7	3	14	14
			yes		9	16	2	3
		≥ 140	no		4	0	13	11
			yes		5	14	4	4

R-U Confidentiality Map



(Duncan, et al. 2004)

R-U Confidentiality Map



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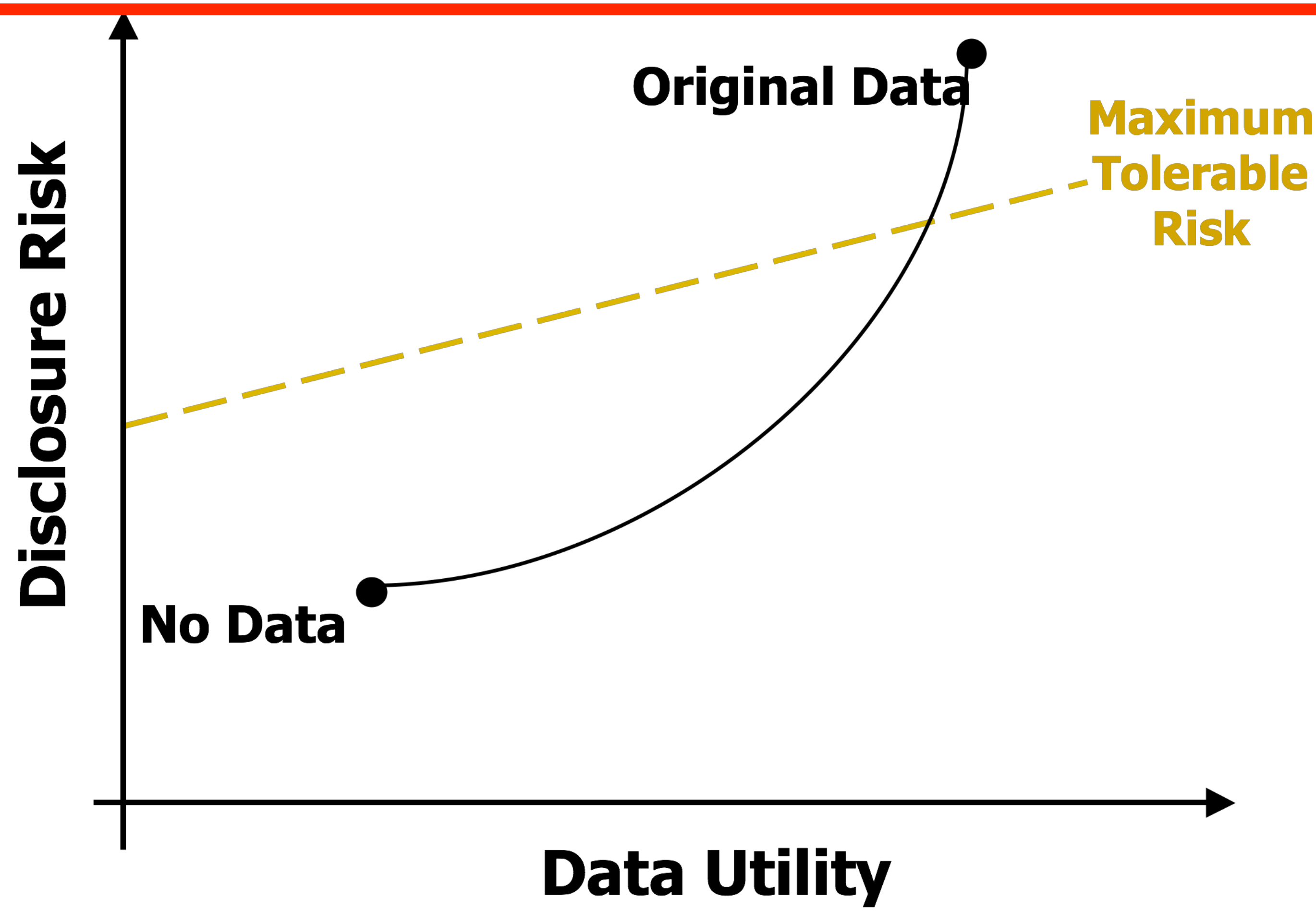
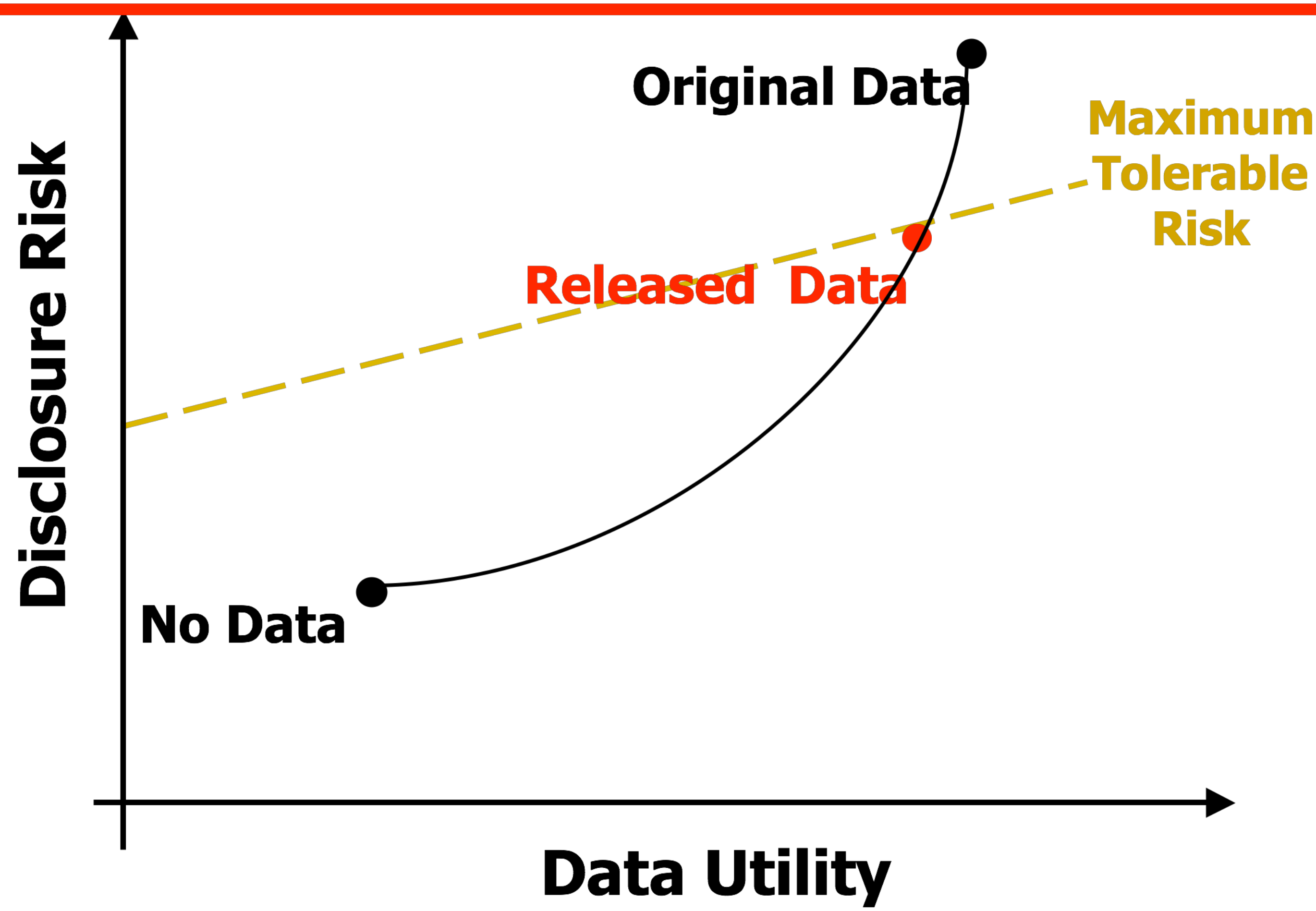
R-U Confidentiality Map



(Duncan, et al. 2004)

R-U Confidentiality Map



(Duncan, et al. 2004)

Disclosure Limitation for Sparse Count Data

- **Uniqueness in population table ⇔ cell count of "1":**
  - **Uniqueness allows intruder to match characteristics in table with other data bases that include same variables to learn confidential information.**
- **Utility typically tied to usefulness of marginal totals:**
  - **Other types of sensible summary statistics?**
- **Risk concerned with small cell counts.**
  - **Assess using bounds given marginal totals.**

Ex. 2: Genetics Linkage

- **Data come from a barley milkdew experiment.**
  - **Edwards (1992). *CDSA*.**
- **37 binary variables (genes) and 81 cases (5% missing data).**
- **Subset of 6 genes that appear closely linked on basis of marginal distributions?**
- **On same chromosome?**

Ex. 2: The Data

			1				2				D
			1		2		1		2		E
			1	2	1	2	1	2	1	2	F
1	1	1	0	0	0	0	3	0	1	0	
		2	0	1	0	0	0	1	0	0	
	2	1	1	0	1	0	7	1	4	0	
		2	0	0	0	2	1	3	0	11	
2	1	1	16	1	4	0	1	0	0	0	
		2	1	4	1	4	0	0	0	1	
	2	1	0	0	0	0	0	0	0	0	
		2	0	0	0	0	0	0	0	0	
A	B	C									

Ex. 3: Australia Census Data

- **10-dimensional highly sparse contingency table extracted from 1981 Australian population census (10 million people):**

Variable	BPL	SEX	AGE	REL	MST	DUR	QAL	INC	FIN	TIS
# Categ.	102	2	11	27	5	62	11	15	16	18

- **892,533,945,600 cells!**

Collapsed Tables

- Collapsed 5-way table with 105,600 cells of which 65% are zero

Variable	BPL	MST	QAL	INC	FIN
# Categ.	8	5	11	15	16

- Collapsed 6-way table with 48,000 cells of which 41% are zero

Variable	BPL	SEX	AGE	REL	MST	QAL
# Categ.	8	2	11	5	5	11

Two-Way Fréchet Bounds

- For 2×2 tables of counts $\{x_{ij}\}$ given the marginal totals $\{x_{1+}, x_{2+}\}$ and $\{x_{+1}, x_{+2}\}$:

$$\begin{array}{cc|c} x_{11} & x_{12} & x_{1+} \\ x_{21} & x_{22} & x_{2+} \\ \hline x_{+1} & x_{+2} & n \end{array}$$

$$min(x_{i+}, x_{+j}) \geq x_{ij} \geq max(x_{i+} + x_{+j} - n, 0)$$

- Interested in multi-way generalizations involving higher-order, overlapping margins.

Multi-way Bounds

- **For decomposable log-linear models:**

$$\text{Expected Value} = \frac{\prod \textit{MSSs}}{\prod \textit{Separators}}$$

- ***Theorem*: When released margins correspond to those of decomposable model:**
  - ***Upper bound:* minimum of values from relevant margins.**
  - ***Lower bound:* maximum of zero, or sum of values from relevant margins minus separators.**
  - **Bounds are sharp.**

Fienberg and Dobra (2000)

2^3 Table Given 2×2 Margins

x_{111}	x_{121}	x_{1+1}
x_{211}	x_{221}	x_{2+1}
x_{+11}	x_{+21}	x_{++1}

x_{112}	x_{122}	x_{1+2}
x_{212}	x_{222}	x_{2+2}
x_{+12}	x_{+22}	x_{++2}

x_{11+}	x_{12+}
x_{21+}	x_{22+}

- Obvious upper and lower bounds for x_{111}
    - Extra upper bound: $x_{111} + x_{222}$

Role of Log-linear Models?

- **For 2×2 case, lower bound is evocative of MLE for estimated expected value under independence:**

$$\hat{m}_{ij} = x_{i+}x_{+j}/n.$$

  - **Bounds correspond to log-linearized version.**
  - **Margins are *Minimal Sufficient Statistics (MSS)*.**

- **In *3*-way table of counts, $\{x_{ijk}\}$, we model logarithms of expectations $\{E(x_{ijk})=m_{ijk}\}$:**

$$\log(m_{ijk}) = u + u_{1(i)} + u_{2(j)} + u_{3(k)} + u_{12(ij)} + u_{13(ik)} + u_{23(jk)}$$

- ***MSS* are margins corresponding to highest order *u*-terms: $\{x_{ij+}\}$, $\{x_{i+k}\}$, $\{x_{+jk}\}$.**

Log-linear Models (cont.)

- **Maximum likelihood estimates (MLEs) are found by setting MSSs equal to their expectations:**

$$\hat{m}_{ij+} = x_{ij+} \quad \text{for } i = 1,2,\ldots,I,\ j = 1,2,\ldots,J,$$

$$\hat{m}_{+jk} = x_{+jk} \quad \text{for } j = 1,2,\ldots,J,\ k = 1,2,\ldots,K,$$

$$\hat{m}_{i+k} = x_{i+k} \quad \text{for } i = 1,2,\ldots,I,\ k = 1,2,\ldots,K.$$

Existence of MLEs for 2×2×2 Table

$0+\delta$	$x_{121}-\delta$	x_{1+1}
$x_{211}-\delta$	$x_{221}+\delta$	x_{2+1}
x_{+11}	x_{+21}	x_{++1}

$x_{112}-\delta$	$x_{122}+\delta$	n_{1+2}
$x_{212}+\delta$	$0-\delta$	n_{2+2}
x_{+12}	x_{+22}	n_{++2}

x_{11+}	x_{12+}
x_{21+}	x_{22+}

Delta must be zero and MLE doesn't exist.

Two Other Three-Way Examples with [12][13][23]

- 3^3 table where MLE exists

3	0	0
0	4	0
0	0	4

0	0	1
5	0	0
0	2	0

0	1	0
0	0	5
3	0	0

- 4^3 table where MLE does not exist

0	0	0	4
0	0	1	2
0	1	2	3
5	1	2	3

4	0	0	2
5	0	5	2
5	6	5	2
1	0	0	0

1	5	0	2
5	3	4	2
0	2	0	0
1	2	0	0

1	5	3	2
0	0	2	0
0	2	4	0
1	2	3	0

Existence of MLEs

- **Linked to pattern of zeros.**
- **Discoverable by defining basis for models and using algebraic and polyhedral geometry.**
- **Examples discovered using algebraic software: *Polymake*.**
- **General theorems in Haberman (1974) and "constructively" in Rinaldo (2005):**
  - **Currently being implemented in C++ and R.**

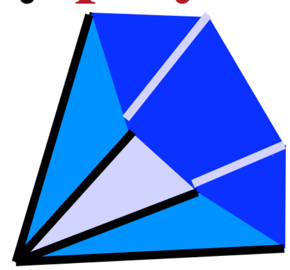
Two Faces of Algebraic Statistics

1. **Conditional Inference**: study and characterization of portions the sample space and, in particular, of all datasets having the observed margins ("exact distribution").
2. **Representation of a Statistical Model**: alternative, more powerful, description of the parameter space.

Its All About Geometry

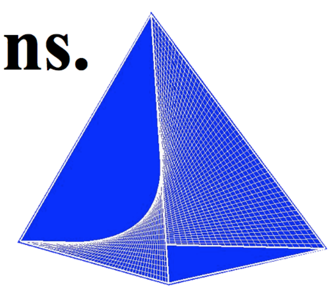
- **Polyhedral Geometry: virtually all data-related quantities can be described by polyhedra.**

Polytope



Polyhedral Cone

- **Algebraic Geometry: a statistical model is specified by a polynomial map. The set of probability distributions is a hyper-surface of points satisfying polynomial equations.**



Algebraic (Toric) Variety

Graphical & Decomposable Log-linear Models

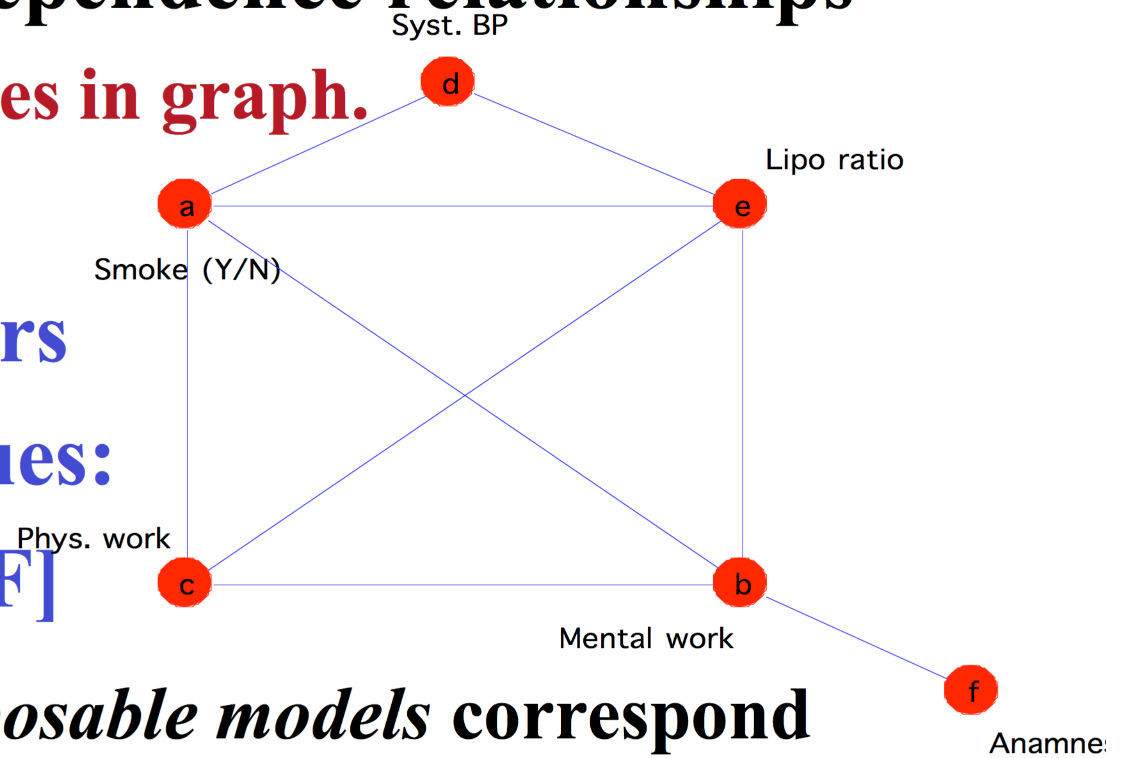
- *Graphical models:* **defined by simultaneous conditional independence relationships**
  - **Absence of edges in graph.**

Example 1:

Czech autoworkers

Graph has 3 cliques:

[ADE][ABCE][BF]



Decomposable models **correspond to triangulated graphs.**

Multi-way Bounds

- **For decomposable log-linear models:**

$$\text{Expected Value} = \frac{\prod \mathit{MSSs}}{\prod \mathit{Separators}}$$

- ***Theorem*: When released margins correspond to those of decomposable model:**
  - ***Upper bound:* minimum of values from relevant margins.**
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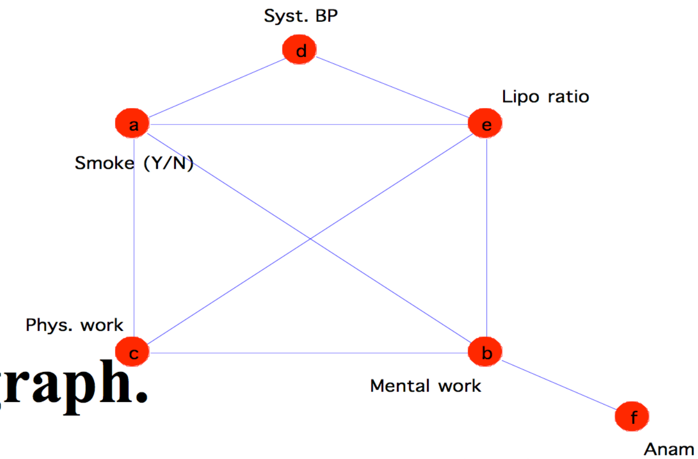
  - **Bounds are sharp.**

Fienberg and Dobra (2000)

Ex. 1: Czech Autoworkers



- **Released margins:**

  [ADE][ABCE][BF]

  - Correspond to decomposable graph.
  - Cell containing population unique has bounds [0, 25].
  - Cells with entry of "2" have bounds: [0,20] and [0,38].
  - Lower bounds are all "0".
- **"Safe" to release these margins; low risk of disclosure.**

Bounds for [BF][ABCE][ADE]

				B	no		yes	
F	E	D	C	A	no	yes	no	yes
neg	< 3	< 140	no		[0,88]	[0,62]	[0,224]	[0,117]
			yes		[0,261]	[0,246]	[0,25]	[0,38]
		≥ 140	no		[0,88]	[0,62]	[0,224]	[0,117]
			yes		[0,261]	[0,151]	[0,25]	[0,38]
	≥ 3	< 140	no		[0,58]	[0,60]	[0,170]	[0,148]
			yes		[0,115]	[0,173]	[0,20]	[0,36]
		≥ 140	no		[0,58]	[0,60]	[0,170]	[0,148]
			yes		[0,115]	[0,173]	[0,20]	[0,36]
pos	< 3	< 140	no		[0,88]	[0,62]	[0,126]	[0,117]
			yes		[0,134]	[0,134]	[0,25]	[0,38]
		≥ 140	no		[0,88]	[0,62]	[0,126]	[0,117]
			yes		[0,134]	[0,134]	[0,25]	[0,38]
	≥ 3	< 140	no		[0,58]	[0,60]	[0,126]	[0,126]
			yes		[0,115]	[0,134]	[0,20]	[0,36]
		≥ 140	no		[0,58]	[0,60]	[0,126]	[0,126]
			yes		[0,115]	[0,134]	[0,20]	[0,36]

Example 1: What to Release?

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- **Among all 32,000+ decomposable models, the tightest possible bounds for three target cells are: (0,3), (0,6), (0,3).**
  - **31 models with these bounds! All involve [ACDEF].**
  - **Another 30 models have bounds that differ by 5 or less and these involve [ABCDE].**

Example 1: What to Release?

- **Among all 32,000+ decomposable models, the tightest possible bounds for three target cells are: (0,3), (0,6), (0,3).**
  - **31 models with these bounds! All involve [ACDEF].**
  - **Another 30 models have bounds that differ by 5 or less and these involve [ABCDE].**
- **Can actually show that release of everything else is "safe": i.e., we can release [ACDE][ABCDF][ABCEF][BCDEF][ABDEF]**

Ex. 2: Genetic Linkage Data

			1				2				D
			1		2		1		2		E
			1	2	1	2	1	2	1	2	F
1	1	1	0	0	0	0	3	0	1	0	
		2	0	1	0	0	0	1	0	0	
	2	1	1	0	1	0	7	1	4	0	
		2	0	0	0	2	1	3	0	11	
2	1	1	16	1	4	0	1	0	0	0	
		2	1	4	1	4	0	0	0	1	
	2	1	0	0	0	0	0	0	0	0	
		2	0	0	0	0	0	0	0	0	
A	B	C									

Ex. 2: Existence of MLEs?

- When we fit model corresponding to [ACD][ADE][ADF][CE][CF][EF][BCD][BDE][BDF]

			1				2				D
			1		2		1		2		E
			1	2	1	2	1	2	1	2	F
1	1	1	0	0	0	0	+	0	+	0	
		2	0	+	0	0	0	+	0	0	
	2	1	+	0	+	0	+	+	+	0	
		2	0	0	0	+	+	+	0	+	
2	1	1	+	+	+	0	+	0	0	0	
		2	+	+	+	+	0	0	0	+	
	2	1	0	0	0	0	0	0	0	0	
		2	0	0	0	0	0	0	0	0	
A	B	C									

Ex. 2: Cont.

- **For [ACD][ADE][ADF][CE][CF][EF][BCD][BDE][BDF] there are 42 problematic zero cells:**
  - **Detected by generalized shuttle algorithm for bounds and verified by MLE software.**
  - **Correspond to zeros in all 255,880 tables.**
  - **Extended MLE exists here.**
- **For no-2nd-order interaction model there are 15 MSS marginals and no problematic zeros.**
  - **Based on shuttle algorithm and verified by MLE software.**
    - **8,628,046 tables.**

Discovering Non-existence Using Bounds

- Replace positive counts by counts of 1.
- Run bounds algorithm and/or LP on 0-1 table.
  - Look for: upper bound = lower bound = 0.
  - Fractional LP bounds may not detect non-existence.
- Compare with methods for detecting non-existence of MLEs.
  - Is bounds software simpler than MLE software?

Degenerate MLE

- **Fixing all 15 positive 3-way margins produces following bounds using integer programming procedure in "*lp solve*":**

A	B	C									
			1				2				D
			1		2		1		2		E
			1	2	1	2	1	2	1	2	F
1	1	1	[0, 1]	[0, 0]	[0, 2]	[0, 0]	[1, 4]	[0, 1]	[0, 2]	[0, 1]	
		2	[0, 0]	[0, 2]	[0, 0]	[0, 2]	[0, 1]	[0, 2]	[0, 1]	[0, 1]	
	2	1	[0, 1]	[0, 0]	[0, 2]	[0, 0]	[6, 9]	[0, 1]	[1, 4]	[0, 1]	
		2	[0, 0]	[0, 1]	[0, 0]	[0, 2]	[0, 1]	[1, 4]	[0, 1]	[9, 12]	
2	1	1	[15, 18]	[0, 1]	[0, 4]	[0, 1]	[0, 1]	[0, 0]	[0, 1]	[0, 0]	
		2	[0, 1]	[2, 5]	[1, 2]	[1, 5]	[0, 0]	[0, 1]	[0, 0]	[0, 1]	
	2	1	[0, 1]	[0, 0]	[0, 2]	[0, 1]	[0, 1]	[0, 0]	[0, 1]	[0, 0]	
		2	[0, 0]	[0, 1]	[0, 1]	[0, 2]	[0, 0]	[0, 1]	[0, 0]	[0, 1]	

Ex. 3: Collapsed Tables

- **Collapsed 5-way table with 105,600 cells of which 65% are zero**

Variable	BPL	MST	QAL	INC	FIN
# Categ.	8	5	11	15	16

- **Collapsed 6-way table with 48,000 cells of which 41% are zero**

Variable	BPL	SEX	AGE	REL	MST	QAL
# Categ.	8	2	11	5	5	11

Ex. 3: 5-way Table

- **Table has 105,600 cells; 65% are 0.**
  - **We set counts in all positive cells = 1 to simplify the problem.**
- **Then we use LP to find upper bounds of cells when all the 2-way margins are fixed.**
  - **We can run the LP solver for the table cells in parallel.**
  - **In our experiment, we used cluster of 64 processors and it took about 4 hours.**
  - **Upper bounds of the cells are all positive, so there are no structural zeros found for this 5-way table.**

Ex. 3: 6-way Table

- **Table has 48,400 cells and 41% have zero cells.**
  - **Use 0-1 representation again.**
  - **Fixed all 2-way margins.**
  - **All upper bounds found are positive–MLEs exist.**
  - **Took about 1 hour on the cluster of 64 processors.**
- **Issue: Can we scale to larger models and bigger tables?**

Summary

- What do we mean by sparseness:
  - Three examples of contingency tables
- Confidentiality & bounds for cell entries
- Existence of MLEs for contingency tables
- Role of computational algebraic geometry
- Exploring linkages between bounds and MLEs
- Undone: Scaling up computations

The End

- **Based in part on paper:**

  A. Dobra, S.E. Fienberg, A. Rinaldo, and Y. Zhou: "Confidentiality Protection and Utility for Contingency Table Data: Algorithms and Links to Statistical Theory."

- **Many related papers available for downloading at**

  http://www.niss.org

  www.stat.cmu.edu/~fienberg/DLindex.html

References

- **Dobra, A. and Fienberg, S. E. (2000). Bounds for cell entries in contingency tables given marginal totals and decomposable graphs. *PNAS*, 97, 11885–11892.**
- **Dobra, A. & Fienberg, S. E. (2003). In *Foundations of Statistical Inference: Proceedings of Shoresh Conference 2000* (Y. Haitovsky, H.R. Lerche, and Y. Ritov, eds.) 3–16.**
- **Eriksson, N., Fienberg, S. E., Rinaldo, A., & Sullivant, S. (2005). Polyhedral conditions for the nonexistence of the MLE for hierarchical log-linear models. *Journal of Symbolic Computation*, 41, 222–233.**
- **Fienberg, S. E. & Rinaldo, A. (2007). Three centuries of categorical data analysis: Log-linear models and maximum likelihood estimation. *JSPI*, 137, 3430-3445.**
- **Rinaldo, A. (2006). On maximum likelihood estimation for log-linear models. Submitted for publication.**

Bounds for *k*-way Table Entries

- LP and IP approaches are NP-hard.
- Develop efficient methods for several special cases, exploiting linkage to statistical theory where possible:
  - Released margins corresponding to decomposable models have explicit formulae.
  - Margins corresponding to reducible graphs can be broken up into smaller problems.
  - Simple result for 2^k tables with release of all (k-1)-dimensional margins fixed.
- Generalized Shuttle algorithm (Dobra, 2001) for residual cases.

2×2 Table: The Data

Observed Counts

x_{11}	x_{12}
x_{21}	x_{22}

Released Margins

$$t = Ax$$

Design Matrix

	x_{11}	x_{12}	x_{21}	x_{22}
$t_1 = x_{1+}$	1	1	0	0
$t_2 = x_{2+}$	0	0	1	1
$t_3 = x_{+1}$	1	0	1	0
$t_4 = x_{+2}$	0	1	0	1

- **Set of all tables having margins *t* are integer points inside a polytope and form the *fiber*:**

$$\{x \in R^4_{\geq 0}, Ax = t\}$$

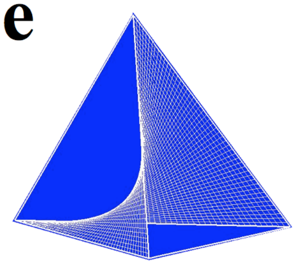
2×2 Table: The Model

p_{11}	p_{12}
p_{21}	p_{22}

- **We are interested in the distribution of the 4 cells in the table specified by the vector of log probabilities:**

$$\log(p_{11}, p_{12}, p_{21}, p_{22}) = A'\theta = (\theta_1 + \theta_3, \theta_1 + \theta_4, \theta_2 + \theta_3, \theta_2 + \theta_4)$$

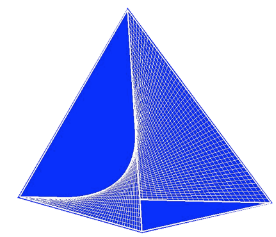
- **The set of all probability distributions for the model of independence need to satisfy one polynomial equation: $p_{11}p_{22}-p_{12}p_{21}=0$, and belong to surface of independence:**



Segre Variet

Design Matrix *A*

MLE



Sample Space

A identifies the fiber:

the set of all tables having the same margins.

$$\{x \geq 0, Ax = t\}$$

Leads to the generalized hypergeometric distribution.

Parameter Space

A specifies the set of polynomial equations that encode the dependence among the variables.

All probability vectors satisfy binomial equations:

$$p^{u+} - p^{u-} = 0$$

all integer u ∈ kernel (A).

Warning: Bounds and Gaps

- **Bounds may not not be sufficient to understand degree of protection for confidentiality.**
  - Gaps in range of values for specific cells are possible!
- **Consider possible 6×4×3 tables:**
  - Specify values for (1,1,1) cell: 0 and 2 (with gap at 1).
  - Can construct margins for which gaps are realized:

2	1	1	0
1	0	0	1
2	2	0	0
0	0	2	2
2	0	2	0
0	2	0	2

2	2	0
1	1	0
2	0	2
3	0	1
0	2	0
0	1	3

2	3	2
2	1	2
2	1	2
2	1	2