Institute for Employment Research

The Research Institute of the Federal Employment Agency

IAB-Discussion Paper 22/2018

Articles on labour market issues

Full and Constrained Pareto Optimality with Incomplete Financial Markets

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ISSN 2195-2663

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Abstract

We study the efficiency properties of equilibria in a neighborhood of economies with Pareto optimal (PO) initial endowments, when the number of agents is finitely large. We provide conditions under which, for all the economies in some open neighborhood of the PO reference economy, all the equilibria are constrained PO (CPO). We also provide conditions under which each open neighborhood of these exceptional PO economies contains both open sets of economies with CPO equilibria and open sets of economies where CPO fails.

Zusammenfassung

Wir untersuchen die Effizienzeigenschaften von Gleichgewichten in einer ökonomischen Umgebung mit Pareto-optimale (PO) Anfangsausstattungen, wenn die Anzahl der Agenten endlich groß ist. Wir zeigen Bedingungen, unter denen für alle Volkswirtschaften in einer offenen Umgebung der PO-Referenzökonomie, alle Gleichgewichte beschränkte PO (CPO) sind. Wir zeigen auch Bedingungen, unter denen jede offene Umgebung dieser außergewöhnlichen PO-Ökonomien sowohl offene Mengen von Ökonomien mit CPO-Gleichgewichten enthält als auch offene Mengen von Ökonomien, in denen CPO versagt.

JEL classification: D51, D52

Keywords: GEI, constrained Pareto inefficiency, numeraire assets.

1 Introduction

The purpose of this paper is to study the efficiency properties of equilibria when financial markets are incomplete. We adopt the notion of constrained efficiency (CPO) introduced in the literature by Geanakoplos and Polemarchakis (1986):[1] an equilibrium allocation is CPO if it is not Pareto inferior to an allocation obtained perturbing portfolios and adjusting commodity prices to restore the equilibrium on these markets. Our focus is on economies close to the ones with a PO equilibrium. In GEI models, this set is exceptional (i.e., it is close and nowhere dense), but it is still of interest for both technical and substantive reasons which we will discuss later on.

When the number of agents is small - compared to the one of non-numeraire commodities - equilibria are generically CP inefficient, see Geanakoplos and Polemarchakis (1986) and Citanna *et alii* (1998). Pareto efficiency of equilibria may occur, but a generic perturbation of the parameters of the economy suffices to restore lack of CPO. When there is a large, but finite, number of agents, neither CPO nor lack of CPO are generic properties.[2] However, the analytical approach pursued in the literature necessarily fails for this class of economies, because it can be applied only if the number of agents is smaller than the one of non-numeraire commodities.

With many agents, the main difficulty in the analysis of CPO is that this is a global property. This means that we need to take into account the allocations associated with each feasible portfolio reallocation. Apart from special cases, it is doubtful that we can reach significant results. There are, however, several classes of economies such that we can sidestep this difficulty because, if a Pareto dominant portfolio exists, it must be arbitrarily close to the equilibrium portfolio. Therefore, a local analysis of the properties of equilibria suffices. This is the case, for instance, for economies sufficiently close to the ones with identical, homothetic state preference. Some results for this class of economies are reported in Mendolicchio and Pietra (2016). Here, we consider economies lying in open neighborhoods of economies with a Pareto optimal equilibrium. We provide two results. Consider an economy with a Pareto optimal equilibrium and more agents (H) than non-numeraire commodities $((S+1)(C-1))$. If the matrix of the excess demand functions satisfies the generic full row rank condition and the volume of trade is sufficiently small, then all the economies in some open neighborhood of this exceptional economy have a unique CPO equilibrium. This follows from two properties: the full row rank condition guarantees that there are no Pareto superior feasible portfolios arbitrarily close to the equilibrium one. Pareto optimality of the equilibrium, and sufficiently small volume of trade, guarantee that, if a Pareto dominating portfolio exists, it must be arbitrarily close to the equilibrium one. This property plays a key role in our argument, since we also show that, for all the economies sufficiently close to one with a Pareto optimal, no-trade equilibrium, CPO equilibria can be obtained as a solution to a well-defined, locally strictly-concave problem, so that the first order conditions of this optimization problem characterize the CPO allocations. On the other hand, if the

[1] See also Stiglitz (1982).

[2] The restriction to economies with a finite number of agents is essential. As reported in Citanna et alii (1998), unpublished work by Mas-Colell (1987) and Kajii (1992) shows that in large economies with well-dispersed characteristics equilibria are typically CPO.

Pareto optimal equilibrium is characterized by no-trade, then each neighborhood of the PO economy contains open sets of economies with a unique CPO equilibrium, and open sets of economies with a unique non CPO equilibrium. Evidently, the set of PO economies with no-trade is lower dimensional with respect to the set of economies with a PO equilibrium. However, all the sets we construct are open, so that the two sets of economies, with and without CPO equilibria, have the same topological dimension.[3] This second result is somewhat counterintuitive. Economies with PO initial endowment are the canonical example of well-behaved economies in terms of regularity and comparative statics. In our framework, they are the intersection in parameter space of paths of economies with CPO equilibria and of other paths of economies with non CPO equilibria. This is made clear by the example concluding the paper. The technical explanation is that, at PO allocations, we have a collapse of the rank of the matrix describing the derivatives of the indirect utility functions with respect to prices: when there is no-trade, this matrix is nil. Arbitrarily small changes in the parameters, i.e., endowments and utilities, may generate matrices spanning completely different subspaces: some perturbations of the matrix, i.e., of the economy, allow for the existence of a Pareto improving price perturbation, other do not allow for them. For economies close to the original, reference, economy with a no-trade PO equilibrium, this is enough to imply, respectively, lack of CPO or CPO of equilibria.

In the next section, we briefly describe the canonical GEI model and our notation. The main results are reported in Section 3, where we also propose three parametric examples constructed for economies in a neighborhood of an economy with a PO allocation. In the first, all the equilibria of economies in some open neighborhood of the original economy are CPO. The second and third show that, in each neighborhood of another economy with a PO allocation, there are open sets of economies with a unique CPO equilibrium and other open sets where the unique equilibrium is not CPO.

2 The Model

The model is a standard two-periods GEI model with numeraire assets. There is a finite set of agents ($h = 1, ..., H$), and a finite set of commodities ($c = 1, ..., C$) at each spot, denoted by $s = 0, ..., S$. Spot $s = 0$ is today, $s > 0$ is a state of the world in the next period. A consumption plan is $x_h \equiv \left(x_h^0, x_h^1, ..., x_h^S\right) \in R_+^{(S+1)C}$, a portfolio is $b_h \equiv \left(b_h^1, ..., b_h^J\right) \in R^J$. Commodity prices are $p \equiv \left(p^0, p^1, ..., p^S\right) \in R_{++}^{(S+1)C}$, asset prices are $q \equiv \left(q^1, ..., q^J\right) \in R^J$. We normalize the price of good 1 at each spot. Asset payoffs are defined in terms of the numeraire commodities and described by a full rank, $(S \times J)$ matrix R with rows in general position,

$$R \equiv \begin{bmatrix} r^{11} & & r^{1J} \\ \vdots & \ddots & \vdots \\ r^{S1} & & r^{SJ} \end{bmatrix}.$$

$Y(q) \equiv \left[-q^T, R^T\right]^T$ is the $((S+1) \times J)$ assets' price-payoffs matrix.

[3] The space of economies is parameterized by endowments and utility functions. Therefore, there is no canonical measure theoretic notion of size.

Finally, $u_h(x_h)$ is agent h's utility function, satisfying the standard assumptions for the differential analysis of equilibria:

Assumption U: For each h, $u_h(x_h)$ is strictly monotone, C^2, differentiably strictly quasi-concave in x_h, and satisfies the boundary conditions: for each $\overline{x}_h \gg 0$, the closure of the set $\{x_h : u_h(x_h) \geq u_h(\overline{x}_h)\}$ is contained in $R^{(S+1)C}_{++}$.

Let $\omega_h \equiv \left(\omega_h^0, \omega_h^1, ..., \omega_h^S\right) \in R^{(S+1)C}_{++}$ be the initial endowment vector. Define $p^s\left(x_h^s - \omega_h^s\right) \equiv p^s \zeta_h^s$ for each s and set $p\zeta_h \equiv \left[p^0\zeta_h^0, ..., p^S\zeta_h^S\right] \in R^{S+1}$.

Consumers' behavior is described as the optimal solution to the problem: Given (p, q),

$$\text{choose } (x_h, b_h) \in \arg\max u_h(x_h) \text{ subject to } p\zeta_h = Y(q)b_h. \qquad (U)$$

Let $\lambda_h \in R^{S+1}_{++}$ be the vector of Lagrange multipliers associated with the optimal solution to problem (U), $V_h(p, q)$ be agent h's indirect utility function, and $\widetilde{V}_h(p, q, \widetilde{b}_h)$ be the $\widetilde{b}_h-$conditional indirect utility function, which associates the maximum attainable level of utility with prices (p, q) and an *exogenously given* portfolio $\widetilde{b}_h$.

We use "$\sim$" to denote functions and variables referred to the $\widetilde{b}-$conditional economy, and the superscript "T" to denote column vectors. Finally, our notation will specify that the demand functions depend upon (ω, u) just when required to avoid possible misunderstandings.

An equilibrium is a price vector $(\overline{p}, \overline{q})$ with associated allocation and portfolio profile $\left\{..., (\overline{x}_h, \overline{b}_h), ...\right\}$ such that:

a. for each h, $(\overline{x}_h, \overline{b}_h)$ solves problem (U) given $(\overline{p}, \overline{q})$,

b. $\sum_h \overline{\zeta}_h = 0$ and $\sum_h \overline{b}_h = 0$.

Given an equilibrium $(\overline{p}, \overline{q})$, and a portfolio $\widetilde{b}$ with $\sum_h \widetilde{b}_h = 0$, a $\widetilde{b}-$conditional equilibrium is a price vector $(\widetilde{p}, \overline{q})$ with allocation $\widetilde{x}$ such that:

c. for each h, $\widetilde{x}_h$ solves problem (U) given $(\widetilde{p}, \overline{q})$ and $\widetilde{b}_h$,

d. $\sum_h \widetilde{\zeta}_h = 0$.

As standard, when testing for the existence of a Pareto superior $\widetilde{b}-$conditional equilibrium, we keep fixed the vector of asset prices at their equilibrium level and, of course, we just consider feasible portfolio reallocations.

We parameterize the set of economies in terms of endowments and utility functions, and we identify the space of economies with $\mathcal{E} \equiv R^{(S+1)CH}_{++} \times \mathcal{U}$. An economy is $(\omega, u) \in \mathcal{E}$, where $R^{(S+1)CH}_{++}$ is endowed with the standard topology, $\mathcal{U}$ with the C^2, compact-open topology, and $\mathcal{E}$ with the product topology, making it into a metric space. Since our results necessarily require perturbations of the utility functions, a set of economies is generic if and only if it is an open and dense subset of $\mathcal{E}$.

By the appropriate version of Walras' law, we can ignore the market clearing conditions for commodity 1 at each spot. Hence, an equilibrium is a zero of the system of the remaining $((S+1)(C-1)+J)$ market clearing equations. We will use $z_h = \left(z_h^0, ..., z_h^S\right) \in$

$R^{(S+1)(C-1)}$ to denote the vector of the excess demand for the non-numeraire commodities.

Let's now formalize the notion of CPO adopted in this paper:

An equilibrium $(\overline{p}, \overline{q})$ is constrained Pareto optimal (CPO) if there is no profile $\widetilde{b} \equiv \left\{ ..., \widetilde{b}_h, ... \right\}$ with $\sum_h \widetilde{b}_h = 0$ such that the associated $\widetilde{b}-$conditional equilibrium $\widetilde{p}$ satisfies $\widetilde{V}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h) \geq V_h(\overline{p}, \overline{q})$, for each h, with at least one strict inequality.

This is also the one adopted in Geanakoplos and Polemarchakis (1986). The only difference is that, in their paper, there is no period zero consumption. In our analysis, we do not impose $\overline{q}\overline{b}_h = \overline{q}\widetilde{b}_h$, for each agent h. This additional restriction would have no effect on our results.

3 Main Results

We now consider the nexus between economies with a Pareto optimal equilibrium allocation and economies with CPO, and non CPO, equilibria. Evidently, under market incompleteness, Pareto efficiency of equilibria holds for an exceptional (i.e., closed and nowhere dense) set of economies. What is of interest here are the constrained efficiency properties of economies which are arbitrarily closed to this exceptional set. As well-known, lack of CPO is a generic property when the number of agents is sufficiently small (i.e., when $H \leq (S+1)(C+1)$). We are going to show that, when $H > (S+1)(C+1)$, there is no unambiguous connection between full PO of the equilibria of an economy and CPO properties of the equilibria of the economies close by.

In economies with a finitely large number of agents, there is a basic difficulty one needs to tackle: CPO (or its lack) for the equilibrium of an economy is a statement on the properties of the entire set of $\widetilde{b}-$conditional equilibria, so that a local analysis does not suffice, in general. However, if we are concerned with equilibria of economies sufficiently close to the one of an economy with a PO equilibrium, local analysis is actually enough, since, as we will establish, a Pareto superior $\widetilde{b}-$conditional equilibrium, if it exists, cannot be bounded away from the actual equilibrium. Even for these sets of economies, there is an additional difficulty, related to the welfare effects of the price adjustments induced by an exogenously given portfolio perturbation. These effects are essential for the analysis, since they are the cause for the possible lack of CPO, but, in general, they make hard to find a simple characterization of CPO (or not CPO) equilibria, even locally. There are two reasonably general sets of economies such that this difficulty can be side-stepped. For economies with state-preferences sufficiently close to be homothetic and identical across consumers, the price adjustments are suitably small. As it turns out, CPO equilibria can be written as the optimal solution to a well-defined, and strictly concave, planning problem, so that the first order conditions of this problem provide a characterization of CPO. This is the approach exploited in Mendolicchio and Pietra (2016), where we go through the details of the argument. A second class of economies such that we can exploit a similar characterization is given by the ones closed to an economy with a *no-trade* PO allocation, as we will establish in Lemma 4.

Let's start with an informal discussion. By the generalization of Roy's Lemma to sequential economies, $\frac{\partial V_h}{\partial p^{sc}} = -\lambda_h^s z_h^{sc}$. Let $\Lambda(\lambda, z)$ be the, normalized, induced $((S+1)(C-1) \times H)$ matrix, for an economy $(\overline{\omega}, \overline{u})$:

$$\Lambda(\overline{\lambda}, \overline{z}) \equiv \begin{bmatrix} -\overline{z}_1^{02} & \cdots & -\overline{z}_H^{02} \\ \vdots & \ddots & \vdots \\ -\frac{\overline{\lambda}_1^S}{\overline{\lambda}_1^0}\overline{z}_1^{SC} & \cdots & -\frac{\overline{\lambda}_H^S}{\overline{\lambda}_H^0}\overline{z}_H^{02} \end{bmatrix}.$$

Evidently, at each PO allocation, $\Lambda(\overline{\lambda}, \overline{z})\,[1, ..., 1]^T = 0$. Consider some open neighborhood of $(\overline{\omega}, \overline{u})$, $B_\delta(\overline{\omega}, \overline{u})$. If, for each $(\omega, u) \in B_\delta(\overline{\omega}, \overline{u})$, there is a strictly positive solution to $\Lambda(\lambda, z)\phi^T = 0$ at its equilibrium, then, by Stiemke's Lemma, there is no price perturbation $\overrightarrow{dp}$ such that $\overrightarrow{dp}\Lambda(\lambda, z) >> 0$. Hence, the equilibrium allocation is, at least locally, CPO. On the other hand, if there is no strictly positive solution to $\Lambda(\lambda, z)\phi^T = 0$, then there is $\overrightarrow{dp}$ such that $\overrightarrow{dp}\Lambda(\lambda, z) >> 0$, so that the equilibrium is definitely not CPO. For the time being, let's just consider the span of $\Lambda(\lambda, z)$. Suppose that $\Lambda(\overline{\lambda}, \overline{z})$ has maximal row rank $(S+1)(C-1)$. To fix ideas, assume that the submatrix given by the first $(S+1)(C-1)$ columns of $\Lambda(\overline{\lambda}, \overline{z})$, call it $\Lambda^{\setminus}(\overline{\lambda}, \overline{z})$ has full rank. Let $\Lambda^{-}(\overline{\lambda}, \overline{z})$ be the matrix given by the its last columns. Then, $\Lambda^{\setminus}(\overline{\lambda}, \overline{z})\,[1]^{\setminus T} = \Lambda^{-}(\overline{\lambda}, \overline{z})\phi^{-T}$, given $\phi^{-} \equiv [1] \in R_{++}^{H-(S+1)(C-1)}$, has a unique solution, $\phi^{\setminus} \equiv [1] \in R_{++}^{(S+1)(C-1)}$. Since $\Lambda^{\setminus}(\overline{\lambda}, \overline{z})$ is a full rank, square matrix, by continuity of the equilibrium, for $(\omega, u) \in B_\delta(\overline{\omega}, \overline{u})$, there is a strictly positive solution $\phi^{\setminus}$ to $\Lambda(\lambda, z)\phi^T = 0$. Hence, the equilibrium can be CPO. Evidently, the argument rests in an essential way on the row rank of $\Lambda(\lambda, z)$ at the reference PO allocation. Whenever the row rank of $\Lambda(\overline{\lambda}, \overline{z})$ is lower than $(S+1)(C-1)$, it is possible that there are open sets of economies in $B_\delta(\overline{\omega}, \overline{u})$ (in fact, in each open set $B(\overline{\omega}, \overline{u})$) such that $\Lambda(\lambda, z)\phi^T = 0$ has no strictly positive solution, which implies that the equilibrium cannot be CPO. This is certainly true when the initial, PO, allocation is no-trade, so that rank $\Lambda(\overline{\lambda}, \overline{z}) = 0$: In this case, each open $B_\delta(\overline{\omega}, \overline{u})$ contains open sets of economies with a unique, CPO equilibrium and other open sets of economies with a unique, non CPO equilibrium.

In the sequel, we will consider only economies close to the ones with a PO no-trade equilibrium. As already mentioned, this guarantees that, locally, the economy has nice concavity properties. Later on, we will add some considerations related to the more general case of economies with a PO equilibrium (with non-zero trade).

We will now make precise the argument outlined above, showing in turn two results: first, for economies sufficiently close to one with a PO equilibrium, if the equilibrium is not CPO, the Pareto dominant portfolio must be arbitrarily close to the portfolio of the reference economy. Hence, in an intuitive sense, local CP optimality entails global CP optimality. This first result is unrelated to the existence, or non-existence, of trade at the equilibrium. We then show our second key result: given any economy with a PO no-trade equilibrium, any of its open neighborhoods contains open sets of economies with a unique CPO equilibrium and open sets of economies with a unique non CPO equilibrium.

The first result follows immediately from the definition of Pareto optimal allocation.

Lemma 3. Let $(\overline{\omega}, \overline{u})$ be any economy with a Pareto optimal equilibrium. Pick any $\xi > 0$.

Then, there is an open neighborhood $B_{\varepsilon}(\overline{\omega},\overline{u})$ such that, for the equilibrium of each $(\omega,u)\in B_{\varepsilon}(\overline{\omega},\overline{u})$, there is no Pareto superior, $\widetilde{b}-$conditional equilibrium with $||\widetilde{b}-b(\omega,u)||>\xi$.

The proof is in Appendix. In view of this Lemma, for economies close to the reference economy $(\overline{\omega},\overline{u})$, each possible Pareto superior portfolio must lie in some bounded neighborhood of the equilibrium portfolio $\overline{b}(\overline{\omega},\overline{u})$. Given that we will now be concerned with no-trade equilibria, this means that $||\widetilde{b}||\leq\delta$, for some $\delta>0$. Without any loss of generality, we will restrict the analysis to feasible portfolios lying in the unit ball. Let

$$\mathbf{S}\equiv\left\{\widetilde{b}\mid||\widetilde{b}||\leq1\right\}\cap\left\{\widetilde{b}\mid\sum_{h}\widetilde{b}_{h}=0\right\}\subset\mathbf{R}^{(H-1)J}.$$

Given any $\mu\in R^{H}_{++}$, define the map $\widetilde{T}_{\mu}(\widetilde{b},\omega,u)$, $\widetilde{T}:\mathbf{S}\times\mathcal{E}\longrightarrow R$, where

$$\widetilde{T}_{\mu}(\widetilde{b},\omega,u)\equiv\sum_{h}\mu_{h}\widetilde{V}_{h}(\widetilde{p}(\widetilde{b},\omega,u),\overline{q}(\omega,u),\widetilde{b},\omega,u),$$

so that $\widetilde{T}_{\mu}(\widetilde{b},\omega,u)$ incorporates both the direct and the indirect effects of a portfolio reallocation on the - weighted - sum of the indirect utilities. $\overline{q}(\omega,u)$ is the vector of equilibrium asset prices of the economy (ω,u), while $\widetilde{p}(\widetilde{b},\omega,u)$ is the vector of $\widetilde{b}-$conditional equilibrium prices given (ω,u). Bear in mind that we are considering economies with an initial endowment close to be PO. Hence, we can assume that equilibria and $\widetilde{b}-$conditional equilibria of these economies are unique, at least for $\widetilde{b}$ sufficiently close to the equilibrium portfolio.

The key step is to observe that, locally, $\widetilde{T}_{\mu}(\widetilde{b},\omega,u)$ is strictly concave.

Lemma 4. Pick any economy $(\overline{\omega},\overline{u})$ such that the Hessian matrix $D^{2}_{x_h}u_h$ is negative definite and the equilibrium is no-trade and Pareto optimal. Fix $\overline{\mu}=\frac{1}{\lambda^{0}}\in R^{H}_{++}$. Then, there is some open set $B_{\delta}(\overline{\omega},\overline{u})$ such that the first order conditions of the problem

$$\max\widetilde{T}_{\overline{\mu}}(\widetilde{b},\omega,u)$$

are necessary and sufficient for a local maximum.

The proof is in Appendix.

Our key result is presented in the following Proposition. Its proof is in Appendix.

Proposition 5. Pick any economy $(\overline{\omega},\overline{u})$ such that the Hessian matrix $D^{2}_{x_h}u_h$ is negative definite and the equilibrium is no-trade and Pareto optimal. Then, for each open set $B_{\delta}(\overline{\omega},\overline{u})$ with $\delta\leq\overline{\delta}$, for some $\overline{\delta}>0$,
i. there is an open set $B^{CPO}\subset B_{\delta}(\overline{\omega},\overline{u})$, such that, for each $(\omega,u)\in B^{CPO}$, the unique equilibrium is CPO,
ii. there is an open set $B^{NCPO}\subset B_{\delta}(\overline{\omega},\overline{u})$, such that, for each $(\omega,u)\in B^{NCPO}$, the unique equilibrium is non CPO.

Remark 6. With trade at the equilibrium, our argument fails. No matter what the net trade profile is, at a Pareto optimal allocation, $\Lambda(\lambda, z)[1]^T = 0$. If $\Lambda(\lambda, z)$ has full row rank, it is easy to check that, locally, there is a vector $\mu >> 0$ such that $\Lambda(\lambda, z)\mu^T = 0$: equilibria may be CPO. They do not have to, because not necessarily $T_{\mu^\circ}(.)$ has nice concavity properties, but they may be CPO. However, the story is quite different when $\Lambda(\lambda, z)$ does not have full row rank at the PO allocation. This is obviously the case at each no-trade equilibrium.

Remark 7. Consider an economy with $H \leq (S+1)(C-1)$ agents. Pick a PO equilibrium. Evidently, the results of Lemma 3 and Lemma 4 still hold, because they do not depend upon the number of agents and commodities. On the other hand, there are no open sets of economies with CPO equilibria. The reason is the following: Consider $\Lambda(\lambda, z)\mu^T = 0$. At the economy with a PO equilibrium, $\Lambda(\overline{\lambda}, \overline{z})\mu = 0$ for $\mu = \left[..., \frac{1}{\lambda_h^0}, ... \right]$. However, since $H \leq (S+1)(C-1)$, $\Lambda(\lambda, z)$ has maximal rank at each equilibrium of a generic set of economies. Hence, at the equilibrium of a generic economy, there is no solution but $\mu = 0$ to $\Lambda(\lambda, z)\mu = 0$, i.e., there are directions $\overrightarrow{d\widetilde{p}}$ such that $\nabla_{\widetilde{p}}\widetilde{V}_h\overrightarrow{d\widetilde{p}} > 0$, for each h. Hence, our argument breaks down and equilibria are, in fact, typically not CPO.

We conclude presenting an extended example. We pick as a reference point an economy $(\overline{\omega}, \overline{u})$ with a PO, no-trade equilibrium. We first show that, for each $\mu >> 0$, the map $\widetilde{T}_\mu(.)$ is strictly concave. In view of Lemma 3, this implies that, for each economy sufficiently close to $(\overline{\omega}, \overline{u})$ the first order conditions for a maximum of $\widetilde{T}_\mu(.)$ are necessary and sufficient, because any conceivable Pareto superior portfolio can be made arbitrarily close to 0.

Next, picking economies with appropriate matrices $\Lambda(\lambda, z)$, we provide

i. an open neighborhood of an economy with a PO equilibrium where all the equilibria are CPO,

ii.a an open set of economies, arbitrarily close to $(\overline{\omega}, \overline{u})$ with a unique CPO equilibrium (since the set is open, for most economies the equilibrium is CPO, but not PO),

ii.b an open set of economies such that the equilibrium is not CPO.

Example 8. The reference economy $(\overline{\omega}, \overline{u})$ is Cobb-Douglas, with two spots in the second period and just one asset, inside money. There are 4 agents, with utility functions

$$u_h(x_h) = \alpha_h^0 \ln x_h^{01} + \left(1 - \alpha_h^0\right) \ln x_h^{02} + \beta_h^1 \left(\alpha_h^1 \ln x_h^{11} + \left(1 - \alpha_h^1\right) \ln x_h^{12}\right) + \beta_h^2 \left(\alpha_h^2 \ln x_h^{21} + \left(1 - \alpha_h^2\right) \ln x_h^{22}\right).$$

The parameters of the economy are described in Table 1.

The $\widetilde{b}-$conditional indirect utility functions are

$$\widetilde{V}_h(.) = \ln\left(\widetilde{p}^0\omega_h^0 - b_h\right) + \left(\frac{1}{2} - \theta_h\right) \ln\left(\widetilde{p}^1\omega_h^1 + b_h\right) + \left(\frac{1}{2} + \theta_h\right) \ln\left(\widetilde{p}^2\omega_h^2 + b_h\right)$$

	ω_h^{01}	ω_h^{02}	ω_h^{11}	ω_h^{12}	ω_h^{21}	ω_h^{22}	a_h^{01}	a_h^{11}	a_h^{21}	β_h^1	β_h^2
$h=1$	$4-a_1^{01}$	$4+a_1^{01}$	$6+2a_1^{11}$	$2-a_1^{11}$	4	4	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$(\frac{1}{2}-\theta)$	$(\frac{1}{2}+\theta)$
$h=2$	9	3	$6-a_2^{11}$	$6+a_2^{11}$	6	6	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$h=3$	2	6	6	2	$2-a_3^{21}$	$6+a_3^{21}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$h=4$	$6+a_1^{01}$	$6-a_1^{01}$	$6-a_1^{11}+a_2^{11}$	$6+a_1^{11}-a_2^{11}$	$9+a_3^{21}$	$3-a_3^{21}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$(\frac{1}{2}+\theta)$	$(\frac{1}{2}-\theta)$

Table 1: Parameters of the economies in Example 8

$$-\left(\alpha_h^{02}\ln p^0+\left(\frac{1}{2}-\theta_h\right)\alpha_h^{12}\ln p^1+\left(\frac{1}{2}+\theta_h\right)\alpha_h^{22}\ln p^2\right).$$

We will consider in detail two vectors of perturbations of the economy: $\theta_2=\theta_3=0$, $\theta_1=-\theta_1$; $a_1^{01}=\theta$, $a_2^{11}=a_3^{21}=\theta^2$ and $a_1^{11}=0$ (and then equilibria are CPO) or $a_1^{11}=-2\theta$ (and then they are not CPO).
By direct computation, the $\widetilde{b}-$conditional spot equilibrium prices (computed for feasible portfolios) are described by:

$$\widetilde{p}^0(\widetilde{b})=\left(1+\frac{\widetilde{b}_2-\widetilde{b}_3}{35}\right),\quad \widetilde{p}^1(\widetilde{b})=\left(1-\frac{\widetilde{b}_1+\widetilde{b}_3}{36-2\theta}\right),\quad \widetilde{p}^2(\widetilde{b})=\left(1+\frac{\widetilde{b}_1+\widetilde{b}_2+2\widetilde{b}_3}{35-2\theta^2}\right).$$

First, let's check that, in a neighborhood of the reference no-trade and PO economy, we can describe CPO allocations as a solution to a concave optimization problem. It is easy to see that each CPO allocation can be described as the optimal solution to the optimization problem

$$\max_{\widetilde{b}} T_\mu(.)=\sum_{h<1}\mu_h\widetilde{V}_h(\widetilde{p}(\widetilde{b}),\widetilde{b}_h)\qquad \text{subject to } \sum_h\widetilde{b}_h=0,$$

for some $\mu>>0$. Moreover, if an allocation is not CPO, it does not solve this maximization problem for any vector $\mu>>0$.

Fix $\mu=\left[...,\frac{1}{\lambda_h^0},...\right]$. The easiest way to check for the concavity of the optimization problem is to compute the second order derivative of $T_\mu(.)$ with respect to $(\widetilde{b},\widetilde{p})$ and then to check the sign of the quadratic form for the feasible directions, the ones with $\sum_h\widetilde{b}_h=0$, and the associated prices. Notice that we consider the $\widetilde{b}-$conditional equilibrium price map as function of $\left[\widetilde{b}_1,...,\widetilde{b}_{H-1}\right]$, i.e., imposing $\sum_h\widetilde{b}_h=0$.

By direct computation, using the formulas obtained in the proof of Lemma 4, at the reference no-trade and PO equilibrium,

$$D^2_{(\widetilde{b},\widetilde{p})}\widetilde{T}_\mu = \begin{bmatrix} -\frac{1}{256} & 0 & 0 & 0 & \frac{1}{128} & -\frac{1}{512} & -\frac{1}{256} \\ 0 & -\frac{1}{864} & 0 & 0 & \frac{1}{576} & -\frac{1}{576} & -\frac{1}{576} \\ 0 & 0 & -\frac{1}{256} & 0 & \frac{3}{256} & -\frac{1}{512} & -\frac{3}{512} \\ 0 & 0 & 0 & -\frac{1}{864} & \frac{1}{288} & -\frac{1}{576} & -\frac{1}{1152} \\ \frac{1}{128} & \frac{1}{576} & \frac{3}{256} & \frac{1}{288} & \frac{35}{384} & 0 & 0 \\ -\frac{1}{512} & -\frac{1}{576} & -\frac{1}{512} & -\frac{1}{576} & 0 & \frac{17}{384} & 0 \\ -\frac{1}{256} & -\frac{1}{576} & -\frac{3}{512} & -\frac{1}{1152} & 0 & 0 & \frac{35}{768} \end{bmatrix}.$$

To compute the quadratic form on the subspace of the feasible perturbations, we restrict its computations to the directions $\overrightarrow{d} \equiv \left[\overrightarrow{d\widetilde{b}}, -\sum_{h<H} d\widetilde{b}_h, -D_{\widetilde{p}}\widetilde{Z}^{-1}D_{\widetilde{b}}\widetilde{Z}\overrightarrow{db}\right]$, where $\overrightarrow{db} \equiv \left[d\widetilde{b}_1, ..., d\widetilde{b}_{H-1}\right]$, i.e., we pre-multiply $D^2_{(\widetilde{b},\widetilde{p})}\widetilde{T}_\mu$ by the vector

$$\overrightarrow{d} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -\frac{1}{36} & \frac{1}{35} \\ 0 & 1 & 0 & -1 & \frac{1}{35} & 0 & \frac{1}{35} \\ 0 & 0 & 1 & -1 & -\frac{1}{35} & -\frac{1}{36} & \frac{2}{35} \end{bmatrix},$$

and post-multiply it by $\overrightarrow{d}^T$. We obtain

$$\overrightarrow{d}\left[D^2_{(\widetilde{b},\widetilde{p})}T_\mu\right]\overrightarrow{d}^T = \frac{1}{17\,418\,240}\begin{bmatrix} -89\,771 & -19\,296 & -25\,727 \\ -19\,296 & -40\,968 & -18\,540 \\ -25\,727 & -18\,540 & -101\,651 \end{bmatrix}.$$

The leading minors of this last matrix have signs $[-,+,-]$. Hence, it is negative-definite and the first order conditions of the optimization problem are necessary and sufficient for a CPO equilibrium. Replace the feasibility constraint into the objective function. Then, the FOCs are given by

$$\left[\mu_h\nabla_{\widetilde{b}_h}\widetilde{V}_h|_{\widetilde{p}} - \mu_H\nabla_{\widetilde{b}_H}\widetilde{V}_H|_{\widetilde{p}}\right] + \left[\sum_h \mu_h\nabla_{\widetilde{p}}\widetilde{V}_h\right]\nabla_{\widetilde{b}_h}\widetilde{p} = 0, \text{ for each } h < H.$$

Given that $\nabla_{\widetilde{b}_h}\widetilde{V}_h|_{\widetilde{p}} = 0$ at each equilibrium, if $\left[\sum_h \mu_h\nabla_{\widetilde{p}}\widetilde{V}_h\right] = 0$, the equilibrium must be CPO.

Let's now consider the matrix $\Lambda(\overline{\lambda},\overline{z})$. When evaluated at the equilibrium of the reference economy, it is given by

$$\Lambda(\lambda,z) = \begin{bmatrix} -\overline{\lambda}_1^{sn}\overline{z}_1^{s2} & -\overline{\lambda}_2^{sn}\overline{z}_2^{s2} & -\overline{\lambda}_3^{sn}\overline{z}_3^{s2} & -\overline{\lambda}_4^{sn}\overline{z}_4^{s2} \\ -\overline{z}_1^{02} & -\overline{z}_2^{02} & -\overline{z}_3^{02} & -\overline{z}_4^{02} \\ -\left(\frac{1}{2}-\theta\right)\overline{z}_1^{12} & -\frac{1}{2}\overline{z}_2^{12} & -\frac{1}{2}\overline{z}_3^{12} & -\left(\frac{1}{2}+\theta\right)\overline{z}_4^{12} \\ -\left(\frac{1}{2}+\theta\right)\overline{z}_1^{22} & -\frac{1}{2}\overline{z}_2^{22} & -\frac{1}{2}\overline{z}_3^{22} & -\left(\frac{1}{2}-\theta\right)\overline{z}_4^{22} \end{bmatrix},$$

and it is trivial at each no-trade equilibrium. We consider three cases:

i. Let $\theta=0$ and pick any a_1^{s2}, a_2^{s2}, a_3^{s2} such that the associated $\Lambda(\overline{\lambda},\overline{z})$ has full rank. Clearly, $\Lambda(\overline{\lambda},\overline{z})[1]^T=0$ and the equilibrium is PO. By continuity, at the equilibrium of each economy in some open neighborhood of $(\overline{\omega},\overline{u})$, $\Lambda(\lambda,z)[\mu]^T=0$. Therefore, $\overrightarrow{dp}\Lambda(\lambda,z)>>0$ has no solution and the equilibrium is CPO.

i*i.a* $a_1^{s2}=(\theta,0,0)$, $a_2^{s2}=\left(0,\theta^2,0\right)$, $a_3^{s2}=\left(0,0,\theta^2\right)$, $a_1^{s2}=(-\theta,-\theta,-\theta)$. Evidently, $\Lambda(\overline{\lambda},\widehat{z})\mu^T=0$ has a strictly positive solution, $\widehat{\mu}=\left(1,\frac{2}{1+2\theta},\frac{2}{1-2\theta},1\right)$. Hence, there is no direction $\overrightarrow{dp}$ such that $\overrightarrow{dp}\Lambda(\overline{\lambda},\widehat{z})>>0$: at each θ sufficiently small, the equilibrium is CPO.

ii.b Let $a_1^{s2}=(\theta,-2\theta,0)$, $a_2^{s2}=\left(0,\theta^2,0\right)$, $a_3^{s2}=\left(0,0,\theta^2\right)$, $a_1^{s2}=\left(-\theta,2\theta-\theta^2,-\theta^2\right)$. The associated matrix is

$$\Lambda(\overline{\lambda},\overline{z})=\begin{bmatrix}-\overline{\lambda}_1^{sn}\overline{z}_1^{s2} & -\overline{\lambda}_2^{sn}\overline{z}_2^{s2} & -\overline{\lambda}_3^{sn}\overline{z}_3^{s2} & -\overline{\lambda}_4^{sn}\overline{z}_4^{s2}\\ \theta & 0 & 0 & -\theta\\ -\left(\frac{1}{2}-\theta\right)2\theta & \frac{1}{2}\theta^2 & 0 & \left(\frac{1}{2}+\theta\right)(2-\theta)\theta\\ 0 & 0 & \frac{1}{2}\theta^2 & -\left(\frac{1}{2}-\theta\right)\theta^2\end{bmatrix}.$$

At the solution of $\Lambda(\overline{\lambda},\overline{z})\mu^T=0$, $\widehat{\mu}_1=\widehat{\mu}_4$ and $\widehat{\mu}_3=\left(1-2\theta^2\right)$. However, $\widehat{\mu}_2=2\theta-7<0$. It follows that there is a direction $\overrightarrow{dp}$ such that $\overrightarrow{dp}\Lambda(\overline{\lambda},\overline{z})>>0$. For instance, $[1,1,1]\,\Lambda(\overline{\lambda},\overline{z})=\left[2,\frac{1}{2},\frac{1}{2},1\right]\theta^2>>0$. To compute the change in portfolios inducing such a price variation, we solve $\overrightarrow{dp}^T=D_{\widetilde{b}}\widetilde{p}\overrightarrow{db}$, i.e., $\overrightarrow{db}=\left[D_{\widetilde{b}}\widetilde{p}\right]^{-1}\overrightarrow{dp}^T$:

$$\begin{bmatrix}d\widetilde{b}_1\\ d\widetilde{b}_2\\ d\widetilde{b}_3\end{bmatrix}=\begin{bmatrix}\frac{35}{2} & 3\theta-54 & \theta^2-\frac{35}{2}\\ \frac{35}{2} & 18-\theta & \frac{35}{2}-\theta^2\\ -\frac{35}{2} & 18-\theta & \frac{35}{2}-\theta^2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}\left(\theta^2+3\theta-54\right)\\ \left(-\theta^2-\theta+53\right)\\ \left(-\theta^2-\theta+18\right)\end{bmatrix}\beta=\overrightarrow{db}(\beta).$$

By construction, the $\widetilde{b}-$conditional equilibrium price vector associated with the (feasible) portfolio $(\overrightarrow{db},\left(\theta^2-\theta-17\right))$ is $\widetilde{p}(\overrightarrow{db})=[(1+\beta),(1+\beta),(1+\beta)]$. We now compute the values of $\widetilde{V}_h(.)$, for each h, replacing the $\widetilde{b}-$conditional spot prices with the map $\widetilde{p}(\overrightarrow{db})$ and using the portfolio perturbation $\overrightarrow{db}(\beta)$. By direct computation, the derivatives of $\widetilde{V}_h(.)$, $h=1,..,4$, with respect to β, evaluated at $\beta=0$, are given by the strictly positive vector $\left[\frac{1}{4},\frac{1}{24},\frac{1}{16},\frac{1}{12}\right]\theta^2$. Hence, the equilibrium is not CPO.
In Figure 1a and 1b, we present the changes of $\widetilde{V}_h(.)$, for $h=1,2,3,4$, in a neighborhood of $\beta=0$, for $\theta=\frac{1}{20}$ and $\theta=\frac{1}{100}$.

The last two parametric examples essentially define two distinct paths, parameterized by θ, in the space of the economies. For all the economies along the path defined in (*ii.a*)*,* and all the ones sufficiently close to them, the unique equilibrium is CPO. For all the economies on the path defined in (*ii.b*)*,* the unique equilibrium is non-CPO. The two path cross at the economy with the PO, no-trade equilibrium. In terms of efficiency properties, this economy is critical. In fact, it is a critical point of the map $D(.)=\sum_k d_k^2$, where k indexes the collection of the determinants of all the submatrices of $\Lambda(\lambda,z)$.

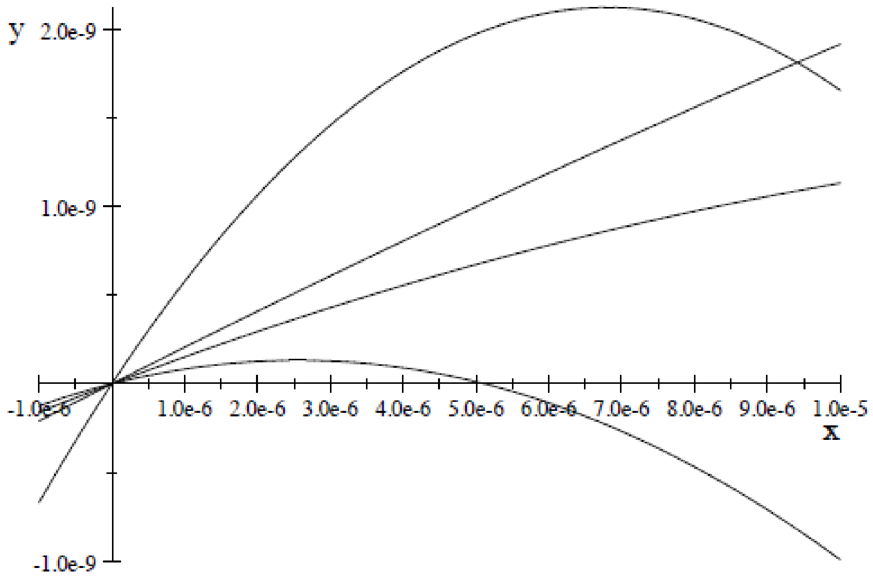

Figure 1a: Changes in the equilibrium utilities, at $\theta = \frac{1}{20}$

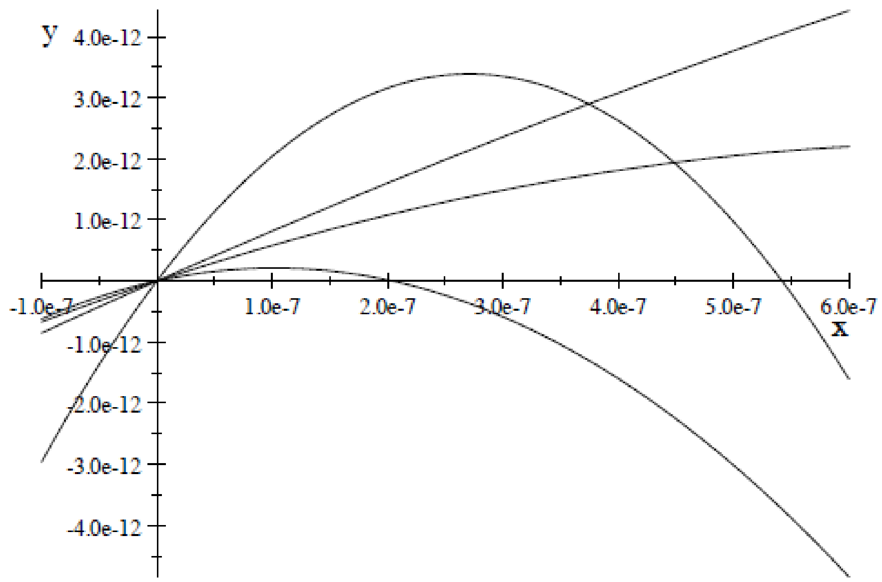

Figure 1b: Changes in the equilibrium utilities, at $\theta = \frac{1}{100}$

4 Conclusions

In this note, we have analyzed CPO of equilibria in GEI with a finitely large number of agents for economies close to one with a PO equilibrium allocation. Suppose that, at this equilibrium, the matrix describing the impact of price changes on the attainable utilities has full rank equal to the number of non-numeraire commodities. Then, for all the economies sufficiently close, the equilibrium is CPO at least with respect to small portfolio perturbations. It is globally CPO if the equilibrium volume of trade at the initial equilibrium is sufficiently small. To the contrary suppose that, at the equilibrium, the same matrix does not have full row rank. Then, each open neighborhood of the reference economy contains open sets of economies with a unique CPO equilibrium and other open sets of economies with a unique, non CPO equilibrium. We formally establish this result for economies where there is no trade at the equilibrium (so that the matrix has rank 0), but it may also hold whenever the row rank of the matrix is not full.

5 Appendix

Proof of Lemma 3: Local uniqueness of equilibria is obvious. Suppose that there is no open $B_\varepsilon(\overline{\omega}, \overline{u})$ with the stated property. Then, we can construct a sequence $\{(\omega^v, u^v)\}_{v=1}^{\infty}$, $(\omega^v, u^v) \to (\overline{\omega}, \overline{u})$ such that, for each v, there is a portfolio profile $\widetilde{b}^v$ such that the associated $\widetilde{b}^v-$conditional equilibrium allocation $\widetilde{x}(\widetilde{b}^v, \omega^v, u^v)$ Pareto dominates the actual equilibrium allocation $x(\omega^v, u^v)$, and such that $||\widetilde{b}^v - b(\omega^v, u^v)|| > \xi$, for each h. Since all sequences can be taken to be convergent, $\widetilde{b}^v \to b^\circ$, $\left(q(\omega^v, u^v), \widetilde{p}(\widetilde{b}^v, \omega^v, u^v)\right) \to \left(q^\circ, \widetilde{p}^\circ\right)$, and $\widetilde{x}^v \to x^\circ$. Moreover, by continuity,

$$\widetilde{V}_h(\widetilde{p}^\circ, \overline{q}, \widetilde{b}_h^\circ, \overline{\omega}, \overline{u}) - V_h(\overline{\omega}, \overline{u}) \geq 0,$$

Since, for each v, $||\widetilde{b}^v - b(\omega^v, u^v)|| > \xi$, it must be $||\widetilde{b}^\circ - b(\overline{\omega}, \overline{u})|| \geq \xi$. Given that R has full rank, this implies $x^\circ \neq \overline{x}$. This is impossible because utility functions are strictly quasi-concave: since $x^\circ \neq \overline{x}$, for each $\pi \in [0,1]$ $x^\pi = \pi\overline{\omega} + (1-\pi)x^\circ$ is a feasible allocation and it is strictly Pareto superior to $\overline{x}$. This contradicts the PO of $\overline{x}$.

Proof of Lemma 4: We are going to show that, locally, the matrix $D^2_{(\widetilde{b},\widetilde{p})}\widetilde{T}_{\mu^\circ}(\widetilde{b}, \omega, u)$ is negative-definite when restricted to the direction $\left[\overrightarrow{db}, D_{\widetilde{b}}\widetilde{p}\overrightarrow{db}\right]$, where $\overrightarrow{db}$ is a feasible portfolio perturbation, i.e., one with $\sum_h d\widetilde{b}_h = 0$, while, by the implicit function theorem,

$$D_{\widetilde{b}}\widetilde{p} = -\left[D_{\widetilde{p}}\widetilde{Z}\right]^{-1} D_{\widetilde{b}}\widetilde{Z},$$

where $\widetilde{Z}(.)$ is the $\widetilde{b}-$conditional aggregate excess demand map. By direct computation, at the equilibrium,

$$D^2_{(\widetilde{b},\widetilde{p})}T_{\mu^\circ}(.)$$

$$= \begin{bmatrix} \begin{bmatrix} \ddots & & \\ & \frac{1}{\lambda_h^0} D_{\widetilde{b}_h} \widetilde{V}(.)|_{\widetilde{p}=\overline{p}} & \\ & & \ddots \end{bmatrix} & -diag\left(\frac{\overline{\lambda}_1^s}{\overline{\lambda}_1^0}\right) \begin{bmatrix} \frac{\partial \widetilde{z}_1^{02}}{\partial \widetilde{b}_1^1} & \cdots & \frac{\partial \widetilde{z}_1^{SC}}{\partial \widetilde{b}_1^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \widetilde{z}_H^{SC}}{\partial \widetilde{b}_H^J} & \cdots & \frac{\partial \widetilde{z}_H^{SC}}{\partial \widetilde{b}_H^J} \end{bmatrix} \\ -diag\left(\frac{\overline{\lambda}_1^s}{\overline{\lambda}_1^0}\right) \begin{bmatrix} \frac{\partial \widetilde{z}_1^{02}}{\partial \widetilde{b}_1^1} & \cdots & \frac{\partial \widetilde{z}_H^{SC}}{\partial \widetilde{b}_H^J} \\ \vdots & \ddots & \vdots \\ \frac{\partial \widetilde{z}_1^{SC}}{\partial \widetilde{b}_1^1} & \cdots & \frac{\partial \widetilde{z}_H^{SC}}{\partial \widetilde{b}_H^J} \end{bmatrix} & -diag\left(\frac{\overline{\lambda}_1^s}{\overline{\lambda}_1^0}\right) \left[D_{\widetilde{p}} \widetilde{Z}(.)\right] \end{bmatrix}.$$

The simple structure of $D^2_{(\widetilde{b},\widetilde{p})} \widetilde{T}_{\mu^\circ}$ follows from two special features of no-trade, Pareto optimal allocations. Let $diag\left(\frac{\overline{\lambda}_1^s}{\overline{\lambda}_1^0}\right)$ be the diagonal, $((S+1)(C-1))$ $-$dimensional matrix with non-zero coefficients given by the normalized state Lagrange multipliers evaluated at the equilibrium. By the generalization of Roy's Lemma to sequential economies, and our choice of the vector μ°, at the equilibrium

$$\nabla_{\widetilde{p}} \widetilde{T}_{\mu^\circ} = -\sum_h diag\left(\overline{\lambda}_h^n\right) \overline{z}_h = -diag\left(\overline{\lambda}_h^n\right) \sum_h \overline{z}_h,$$

because, by PO, the vectors λ_h^n of normalized Lagrange multipliers are agent-invariant. Since the allocation is no-trade, we can ignore all the terms $\frac{\partial \widetilde{\lambda}_h^{sn}}{\partial \widetilde{b}_h^j} \overline{z}_h^{sc}$. Hence, the top right (and bottom left) submatrices have the simple structure reported above. Consider now the quadratic form

$$\begin{bmatrix} \overrightarrow{db} & -\overrightarrow{db} D_{\widetilde{b}} \widetilde{Z}^T \left[D_{\widetilde{p}} \widetilde{Z}\right]^{-1} \end{bmatrix} \left[D^2_{(\widetilde{b},\widetilde{p})} \widetilde{T}_{\mu^\circ}\right] \begin{bmatrix} \overrightarrow{db} \\ -\left[D_{\widetilde{p}} \widetilde{Z}\right]^{-1} D_{\widetilde{b}} \widetilde{Z} \overrightarrow{db} \end{bmatrix}$$

$$= \sum_h \frac{1}{\lambda_h^0} \overrightarrow{db}_h D^2_{\widetilde{b}_h} \widetilde{V}_h \overrightarrow{db}_h + diag\left(\lambda^n\right) \overrightarrow{db} D_{\widetilde{b}} \widetilde{Z}^T \left[D_{\widetilde{p}} \widetilde{Z}\right]^{-1} \left[D_{\widetilde{b}} \widetilde{Z}\right] \overrightarrow{db} \leq 0,$$

with strict inequality for each $\overrightarrow{db} \neq 0$. The last inequality holds because preferences are strictly-concave, by assumption. Since there are no income effects, the matrix $D_{\widetilde{p}} \widetilde{Z}$ is negative-definite. Thus, its inverse is also negative definite. The inequality follows immediately.

Proof of Proposition 5: Given Lemma 4, by regularity of the equilibrium map and of the map $\widetilde{T}_\mu(.)$, there is an open neighborhood of $((\overline{\omega}, \overline{u}), 0)$, let's say $B_\delta(\overline{\omega}, \overline{u}) \times B_\delta(0)$ such that, for each $\overline{\mu}$ close to $\left[\frac{1}{\lambda^0}\right]$, the quadratic form $\left[D^2_{(\widetilde{b},\widetilde{p})} \widetilde{T}_\mu\right]$ evaluated at the equilibrium is negative-definite in each direction $\left[\overrightarrow{db}, D_{\widetilde{b}} \widetilde{p}(\omega, u) \overrightarrow{db}\right]$ for each economy $(\omega, u) \in B_\delta(\overline{\omega}, \overline{u})$ and each $\widetilde{b} \in B_\delta(0)$. Taking into consideration Lemma 3, without loss of generality, we can assume that, for $(\omega, u) \in B_\delta(\overline{\omega}, \overline{u})$, if a Pareto superior $\widetilde{b}-$conditional equilibrium exists, then the portfolio profile satisfies $\widetilde{b} \in B_\delta(0)$.

i. To establish the first result, given $(\overline{\omega}, \overline{u})$, it is sufficient to perturb the initial endow-

ment, using a standard procedure, obtaining a new economy $(\omega^{\circ}, \overline{u})$ such that:
a. the equilibrium price vector $(\overline{p}, \overline{q})$ is not affected,
b. the matrix $\Lambda\left(\overline{\lambda}, z^{\circ}\right)$ evaluated at the equilibrium of $(\omega^{\circ}, \overline{u})$ has full row rank $(S+1)(C-1)$.
Pareto optimality of the equilibrium allocation $x^{\circ} = \overline{\omega}$ implies that $\Lambda\left(\overline{\lambda}, z^{\circ}\right)[1]^{T} = 0$, i.e., $\Lambda^{\backslash}\left(\overline{\lambda}, z^{\circ}\right)[1]^{\backslash T} = \Lambda^{-}\left(\overline{\lambda}, z^{\circ}\right)[1]^{-T}$, where $\Lambda^{\backslash}\left(\overline{\lambda}, z^{\circ}\right)$ is a square $(S+1)(C-1)$-dimensional matrix of full rank, while $\Lambda^{-}\left(\overline{\lambda}, z^{\circ}\right)$ is given by the residual columns of $\Lambda\left(\overline{\lambda}, z^{\circ}\right)$.
By continuity, for each economy (ω, u) in some open neighborhood of $(\omega^{\circ}, \overline{u})$, there is $\mu >> 0$ such that $\Lambda(\lambda, z)\mu^{T} = 0$.
The proof is by contradiction. Consider an open ball $B_{\eta}(\omega^{\circ}, \overline{u}) \subset B_{\delta}(\overline{\omega}, \overline{u})$. Suppose that the unique equilibrium of any $(\widehat{\omega}, \widehat{u}) \in B_{\eta}(\omega^{\circ}, \overline{u})$ is not CPO. Then, there exists some $\widetilde{b} \in B_{\delta}(0)$ such that the associated $\widetilde{b}-$conditional equilibrium allocation $\widetilde{x}$ is Pareto superior to the equilibrium of $(\widehat{\omega}, \widehat{u})$, $\widehat{x}$. Let $\overrightarrow{db} \equiv \left[\widetilde{b} - \widehat{b}\right]$ be the associated feasible portfolio perturbation. Pick $\widehat{\mu} >> 0$ such that $\Lambda(\widehat{\lambda}, \widehat{z})\widehat{\mu}^{T} = 0$. As established above, such a $\widehat{\mu}$ exists. Since $\widetilde{x} \succ_{PO} \widehat{x}$, for each $\mu >> 0$, $\widetilde{T}_{\widehat{\mu}}(\widetilde{b}, \widehat{\omega}, \widehat{u}) > \widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u})$. Using a second order Taylor expansion,

$$
\begin{aligned}
\widetilde{T}_{\widehat{\mu}}(\widetilde{b}, \widehat{\omega}, \widehat{u}) \quad = \quad & \widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u}) + \nabla_{\widetilde{b}}\widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u})\overrightarrow{db} - \nabla_{\widetilde{p}}\widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u})\left[D_{\widetilde{p}}\widetilde{Z}\right]^{-1} D_{\widetilde{b}}\widetilde{Z}\overrightarrow{db} \\
& + \frac{1}{2}\xi\left[D^{2}_{(\widetilde{b}, \widetilde{p})}\widetilde{T}_{\widehat{\mu}}(.)\right]\xi^{T},
\end{aligned}
$$

where $\xi \equiv \left[\overrightarrow{db}, -\overrightarrow{db}D_{\widetilde{b}}\widetilde{Z}^{T}\left[D_{\widetilde{p}}\widetilde{Z}\right]^{-1}\right]$, while $\left[D^{2}_{(\widetilde{b}, \widetilde{p})}\widetilde{T}_{\widehat{\mu}}(.)\right]$ is evaluated at $\widehat{b} + \theta\overrightarrow{db}$, for some $\theta \in [0, 1]$. Evidently, at each equilibrium, $\nabla_{\widetilde{b}}\widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u}) = 0$. By our choice of $\widehat{\mu}$, $\nabla_{\widetilde{p}}\widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u}) = 0$. Since ξ is induced by a feasible portfolio perturbation, $\xi\left[D^{2}_{(\widetilde{b}, \widetilde{p})}\widetilde{T}_{\widehat{\mu}}(.)\right]\xi^{T} < 0$. Hence, $\widetilde{T}_{\widehat{\mu}}(\widetilde{b}, \widehat{\omega}, \widehat{u}) < \widetilde{T}_{\widehat{\mu}}(\widehat{b}, \widehat{\omega}, \widehat{u})$. This contradicts the assumption that the $\widetilde{b}-$conditional equilibrium allocation is Pareto superior to the equilibrium allocation $\widehat{x}$. Hence, the equilibrium is CPO.
ii. This second result is essentially established in Prop. 11, in Mendolicchio and Pietra (2016).

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Imprint

IAB-Discussion Paper 22/2018
12 November 2018

Editorial address
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ISSN 2195-2663

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