Institute for Employment Research

The Research Institute of the Federal Employment Agency

IAB-Discussion Paper 12/2016

Articles on labour market issues

A re-examination of constrained Pareto inefficiency in economies with incomplete markets

Concetta Mendolicchio
Tito Pietra

ISSN 2195-2663

A re-examination of constrained Pareto inefficiency in economies with incomplete markets

Mendolicchio Concetta (Institute for Employment Research - IAB)
Pietra Tito (Università di Bologna - D.S.E.)

Contents

Abstract

We establish that, when the number of agents is sufficiently large, but finite, there are open sets of economies with constrained Pareto inefficient equilibria, and provide a simple sufficient condition for CP inefficiency. We also show that there are open sets of economies with CPO equilibria.

Zusammenfassung

Wir zeigen, dass, wenn die Zahl der Agenten ausreichend groß, aber endlich ist, offene Sets an Ökonomien mit eingeschränkter Pareto-Ineffizienz (CP) existieren und bieten eine einfache hinreichende Bedingung für CP-Ineffizienz. Wir zeigen auch, dass offene Sets an Ökonomien mit CPO-Gleichgewichten existieren.

JEL classification: D51, D52

Keywords: GEI, constrained Pareto inefficiency, numeraire assets.

1. Introduction

With incomplete financial markets, equilibrium allocations are typically Pareto inefficient.[1] The natural question is if they satisfy weaker notions of efficiency, defined taking into account the restrictions that market incompleteness imposes upon the set of feasible allocations. The canonical criterion of constrained Pareto optimality (CPO) has been introduced by Stiglitz (1982) and Geanakoplos and Polemarchakis (1986), and further developed by Citanna, et al. (1998).[2] The key idea is that a minimal efficiency requirement for an allocation is that it should be impossible to Pareto improve by rearranging portfolios, and letting commodity prices adjust to restore market clearing for the commodities. The possibility to attain a Pareto improvement using this limited set of instruments rests on the welfare effects of the induced changes in equilibrium prices.

In this, and a companion paper, Mendolicchio and Pietra (2016), we reconsider the issue of CPO, providing some new results which complement the ones in the literature. Here, we focus on Pareto improvements attainable via portfolio reallocations, adopting the canonical criterion of CPO. In Geanakoplos and Polemarchakis (1986), lack of CPO is established for economies where the number of agents, H, is smaller than the number of normalized commodity prices, $(S+1)(C-1)$. Here, we extend the analysis to economies where the number of agents is finite,[3] but this upper bound fails. The logic of the results in the literature implies that, no matter what the - finite - number of agents is, there are open sets of economies with non CPO equilibria. Think, for instance, of replica economies: If the equilibrium is not CPO with one agent per type, the same equilibrium is also not CPO for each number of replicas, and for each economy sufficiently close to a replica economy. Therefore, there are always open sets of economies with non CPO equilibria, provided that there is - in a proper sense - not that much of heterogeneity across agents.

Our results follow from a completely different logic and hold for all the profiles of utility functions satisfying a mild condition. They can be summarized as follows: consider an economy with a finite, but large, number of agents. Pick any of its equilibria. Fix the equilibrium price and allocation and consider the set of economies with the same total resources and characterized by endowment profiles such that the prices and allocation we started with are also an equilibrium given the new endowment profile. If, at the equilibrium allocation, the matrix of income effects has full rank, then there is a relatively open neighborhood of endowments in the given set such that the equilibrium we started with is not CPO. The result does not depend in any way upon the degree of heterogeneity of the agents. In this specific sense, lack of CPO at equilibrium in GEI is a common phenomenon, independently of the number of agents. Unfortunately, with many agents, it cannot be a generic property, since we also show that there are open sets of economies with CPO equilibria. It follows that neither CPO, nor lack of CPO, are generic properties, independently of any robust restriction on the class of preferences, or of the degree of heterogeneity across agents. Lack of CPO can be established using a purely local argument, as in the previous papers on this topics. To the contrary, to show that there are open sets of economies with CPO equilibria, we need to provide a global argument, taking into account the equilibria associated with each possible reallocation of asset portfolios. This can be accomplished once we restrict the analysis to some set of economies sufficiently "close" to economies with identical,

[1] The set of equilibrium allocation itself may be Pareto ranked, completely, as in the Hart (1975) example, or partially, as in Pietra (2004) and Salto and Pietra (2013), which consider economies with nominal asset and indeterminate equilibria.

[2] See also Nagata (2005) and Tirelli (2008).

[3] The restriction to a finite number of agents is essential. In large economies, our argument of proof does not apply. More important, as reported in Citanna et al. (1998), unpublished papers by Mas-Colell (1987) and Kajii (1992) have shown that equilibria are CPO in large economies with well-dispersed agents.

homothetic preferences. We also provide a simple sufficient condition for the lack of CPO of equilibria. For a generic set of economies, equilibria are CP inefficient when the, properly discounted, present value of the vector of net trades in the numeraire commodities is strictly positive for each agent. This condition is easy to check, once an equilibrium is given. Its weakness is that it is based on both "observables", the net trades, and "non-observables", the normalized vectors of Lagrange multipliers that we need to discount. While it is possible that more satisfactory sufficient conditions could be found, they must all share this shortcoming.

Citanna et al. (1998) prove generic lack of CPO independently of the number of agents. However, they allow for both portfolio and period zero endowment reallocations. Their result is certainly important, but it exploits both the direct welfare effects of the endowment redistribution and the pecuniary externalities generated by endowment and portfolio reallocations. We think that it is interesting to consider what happens in economies with many agents when we have only pecuniary externalities due to portfolio reallocations, i.e., when the possibility of a PO improvement only rests upon the welfare effects of the induced price changes.

Contrary to most of the previous papers on this topics, in studying economies with CP inefficient equilibria, we pursue an approach based on the characterization of the CPO allocations as solutions to a well-defined optimization problem built upon the agents' indirect utility functions.[4] Using the terminology of Citanna et al. (1998), we follow an optimization approach, while both they and Geanakoplos and Polemarchakis (1986) adopt a submersion approach. We believe that it is interesting to fully and explicitly pursue also our approach, since, for certain purposes, it is somewhat more transparent, interpretation-wise.

To extend the analysis of CPO in GEI to economies with many agents is important. Since we are dealing with competitive economies, to impose upper bounds on their number is a very strong restriction. When we get rid of it, the CP inefficiency results become weaker. However, we think that they are still interesting for several reasons. First, we establish that lack of CPO is, while non generic, still a pervasive phenomenon, and that it may hold for any degree of heterogeneity across agents. Secondly, our sufficient condition for lack of CPO is easy to check, once an equilibrium is given. Third, our results make transparent that the same equilibrium allocation, given preferences, may or may not be CPO depending upon the endowment vector. Indeed, for each equilibrium, there is a polyhedron of initial endowments such that the given price and allocation are an equilibrium. The same allocation may be CP inefficient for some initial endowments, and CPO for others.

The next section presents the model and establishes the, fairly standard, properties of equilibria to be exploited later on. In Section 3, we make precise the notion of CPO and prove our main results. We also provide an example of an economy with a non CPO equilibrium: the equilibrium allocation satisfies the sufficient conditions for CP inefficiency presented here, while the economy violates the assumptions required for the Geanakoplos and Polemarchakis (1986) result to apply.

2. The Model

Consider a standard two-periods GEI model with numeraire assets. There is a finite set of agents ($h = 1, ..., H$), and a finite set of commodities ($c = 1, ..., C$) at each of the $(S+1)$ spots, $s = 0, ..., S$. A consumption plan is $x_h \equiv \left(x_h^0, x_h^1, ..., x_h^S\right) \in \mathbb{R}_+^{(S+1)C}$, a portfolio is

[4] Our approach is somewhat in the same vein of Stiglitz (1982).

$b_h \equiv \left(b_h^1, ..., b_h^J\right) \in \mathbb{R}^J$. Commodity prices are $p \equiv \left(p^0, p^1, ..., p^S\right) \in \mathbb{R}_{++}^{(S+1)C}$, asset prices are $q \equiv \left(q^1, ..., q^J\right) \in \mathbb{R}^J$. As usual, we normalize the price of good 1 in each spot. Asset payoffs are defined in terms of the numeraire commodities and described by a $(S \times J)$ full rank matrix R in general position,

$$R \equiv \begin{bmatrix} r^{11} & & r^{1J} \\ \vdots & \ddots & \vdots \\ r^{S1} & & r^{SJ} \end{bmatrix}.$$

$Y(q) \equiv \left[-q^T, R^T\right]^T$ is the assets' price-payoffs matrix.

Finally, $u_h(x_h)$ is agent h's utility function, satisfying the standard assumptions for the differential analysis of equilibria.

Assumption U. For each h, $u_h(x_h)$ is strictly monotone, C^3, differentiably strictly quasi-concave in x_h,[5] and satisfies the boundary conditions: for each $\overline{x}_h \gg 0$, the closure of the set $\{x_h : u_h(x_h) \geq u_h(\overline{x}_h)\}$ is contained in $\mathbb{R}_{++}^{S(C+1)}$.

Consumers' behavior is described as the optimal solution to the problem: Given (p, q),

$$\text{choose } (x_h, b_h) \in \arg\max u_h(x_h) \text{ subject to } p(x_h - \omega_h) \equiv p\zeta_h = Y(q)b_h \qquad (U)$$

where $\omega_h \equiv \left(\omega_h^0, \omega_h^1, ..., \omega_h^S\right) \in \mathbb{R}_{++}^{(S+1)C}$ is the initial endowment vector, while $p\zeta_h \equiv \left[p^0\zeta_h^0, ..., p^S\zeta_h^S\right]^T$. Let $\lambda_h \in \mathbb{R}_{++}^{S+1}$ be the vector of Lagrange multipliers associated with the optimal solution to problem (U).

Also, let $V_h(p, q)$ be agent h's indirect utility function, and $\widetilde{V}_h(p, q, \widetilde{b}_h)$ be the $\widetilde{b}_h-$conditional indirect utility function. This is the function which associates with prices and a given portfolio $\widetilde{b}_h$ the maximum level of utility. Whenever any ambiguity could arise, we will use "$\sim$" to denote functions and variables referred to the $\widetilde{b}-$conditional optimal behavior. Also, if convenient, we will use a superscript "T" to denote column vectors. Finally, our notation will specify that utility, or demand, functions depend upon (ω, u) just when required to avoid possible misunderstandings.

Definition 1. An equilibrium is a price vector $(\overline{p}, \overline{q})$ with associated allocation and portfolio profile $\left\{..., (\overline{x}_h, \overline{b}_h), ...\right\}$ such that:
a. for each $h, (\overline{x}_h, \overline{b}_h)$ solves problem (U) given $(\overline{p}, \overline{q})$,
b. $\sum_h \overline{\zeta}_h = 0$ and $\sum_h \overline{b}_h = 0$.

We parameterize the set of economies in terms of endowments and utility functions, and identify the space of economies with $\mathcal{E} \equiv \mathbb{R}_{++}^{(S+1)CH} \times \mathcal{U}$. An economy is $(\omega, u) \in \mathcal{E}$, where $\mathbb{R}_{++}^{(S+1)CH}$ is endowed with the standard topology, $\mathcal{U}$ with the C^3, compact-open topology, and $\mathcal{E}$ with the product topology. It is well-known that this is a metric space. Since our results necessarily require perturbations of the utility functions, a set of economies is generic if it is an open and dense subset of $\mathcal{E}$, as usual.

By the appropriate version of Walras' law, we can ignore the market clearing conditions for commodity 1 at each spot. Hence, an equilibrium is defined as a zero of the system of $((S+1)(C-1)+J)$ market clearing equations $\Phi(p, q) = 0$. From now on, excess demand

[5] We need utility functions to be C^3 because, in the proof of Prop. 12, we use the second order derivatives of the conditional equilibrium prices with respect to portfolio profiles.

functions for commodities must always be interpreted as $(S+1)(C-1)$ vector valued functions and denoted $z_h(.) \in \mathbb{R}^{(S+1)(C-1)}$, i.e., we will, unless otherwise specified, ignore the excess demand for the numeraire commodities.

For future reference, we need to consider only equilibria satisfying some specific properties, in addition to regularity. Thm. 2 describes them and shows that they are generic.

Let $D_{\widetilde{b}_h} \widetilde{z}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h)$ be the $(S+1)(C-1) \times J$ matrix describing the derivatives of agent h's consumption of commodities $\{2, ..., C\}$ at each spot with respect to asset holdings.

Theorem 2. If $S > J$, there is an open and dense set $\mathcal{E}^r \subset \mathcal{E}$, such that, for each $(\omega, u) \in \mathcal{E}^r$:
i. there is a finite number of equilibria and each equilibrium is regular, i.e., $rank\left[D_{(p,q)}\Phi(p,q)\right] = (S+1)(C-1) + J$, and strongly regular, i.e., $rank\left[D_{\widetilde{p}} \sum_h \widetilde{z}_h(.)\right] = (S+1)(C-1)$, at $\widetilde{b}_h = b_h(\overline{p}, \overline{q})$, for each h,
ii. at each equilibrium, the matrix $Y(\overline{q})$ is in general position,
iii. at each equilibrium $(\overline{p}, \overline{q})$, $(\overline{x}, \overline{b}, \overline{\lambda})$, the $(S+1)(C-1) \times H$ dimensional matrix

$$\Lambda(\lambda, z) \equiv - \left[\begin{array}{ccc} \left[\frac{1}{\overline{\lambda}_1^0} \nabla_{\widetilde{p}} \widetilde{V}_1\right]^T & \cdots & \left[\frac{1}{\overline{\lambda}_1^0} \nabla_{\widetilde{p}} \widetilde{V}_H\right]^T \end{array}\right] = \left[\begin{array}{ccc} -\overline{z}_1^{02} & \cdots & -\overline{z}_H^0 \\ \vdots & \ddots & \vdots \\ -\frac{\overline{\lambda}_1^S}{\overline{\lambda}_1^0} \overline{z}_1^{SC} & \cdots & -\frac{\overline{\lambda}_H^S}{\overline{\lambda}_H^0} \overline{z}_H^{SC} \end{array}\right]$$

has maximal rank, $\min\{(S+1)(C-1), H\}$,
iv. at each equilibrium, if $(H-1)J \geq (S+1)(C-1)$, the matrix

$$\left[..., \left[D_{\widetilde{b}_h} \widetilde{z}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h) - D_{\widetilde{b}_1} \widetilde{z}_1(\widetilde{p}, \overline{q}, \widetilde{b}_1)\right]^T, ...\right]$$

has full row rank at $\widetilde{p} = \overline{p}$ and $\widetilde{b}_h = b_h(\overline{p}, \overline{q})$, for each h.

The proof is in Appendix. Bear in mind that iv. immediately rules out economies with identical, homothetic preferences, while iii. rules out, among others, economies with no-trade equilibria, and, when $H \leq (S+1)(C-1)$, with Pareto optimal equilibrium allocations. Property iv. is required to obtain that portfolio reallocations induce a sufficiently rich set of perturbations of the equilibrium prices. The condition only depends upon the specification of the equilibrium prices and commodity allocation, while it does not depend directly upon (ω, b). Specifically: consider two profiles ω' and ω" such that, at some associated vectors b' and b", the same profile $(\overline{x}, (\overline{p}, \overline{q}))$ is an equilibrium. Then, the matrices in iv. are identical, since they just depend upon the matrix of the income effects, computed at the equilibrium allocation, and the asset payoffs. This fact will play a crucial role later on. In particular, this rank condition implies that, by choosing appropriately $\overrightarrow{db}$, we can obtain each possible profile of commodity price adjustments.

Let's now consider economies with fixed resources K and utility function profile $\overline{u}$. Let

$$\Omega(K) \equiv \left\{\omega \in \mathbb{R}_{++}^{(S+1)CH} | K = \sum_h \omega_h\right\}.$$

For each profile $\omega \in \Omega(K)$, define the set of equilibrium prices, allocations and portfolio profiles as

$$E(\omega) \equiv \{((p,q), (x,b)) \,|(p,q), (x,b) \text{ is an equilibrium of } (\omega, \overline{u})\}.$$

Its projection on $\mathbb{R}_{++}^{(S+1)CH}$ is the set of endowments associated with a no-trade equilibrium,

$$\Omega^{NT}(K) \equiv \{\omega \in \Omega(K) |\, ((x,b) = (\omega, 0), (p,q)) \in E(\omega) \text{ for some } (p,q)\}.$$

Given $\overline{u}$, $\Omega^{NT}(K)$ includes the set of Pareto optimal endowment profiles, and, of course, it includes many other ω, when markets are incomplete. Given $\overline{\omega} \in \Omega^{NT}(K)$, $E^{-1}(\overline{\omega}, 0, p(\overline{\omega}), q(\overline{\omega}))$ defines the set of ωs supporting the same no-trade equilibrium. Evidently, $E^{-1}(\overline{\omega}, 0, p(\overline{\omega}), q(\overline{\omega}))$ is defined by the system of linear equations and inequalities

$$\begin{aligned} \text{for each } h \quad &: \quad p^0(\overline{\omega})\overline{\omega}_h^0 = p^0(\overline{\omega})\omega_h^0 - qb_h, \text{ and} \\ p^s(\overline{\omega})\overline{\omega}_h^s \quad &= \quad p^s(\overline{\omega})\omega_h^s + r^s b_h, \text{ for each } s > 0, \\ \sum_h \omega_h \quad &= \quad K, \quad \omega >> 0. \end{aligned}$$

In our set-up, $E^{-1}(\overline{\omega}, 0, p(\overline{\omega}), q(\overline{\omega}))$ is the counterpart of the set of endowment points lying on the tangent to the indifference curves though any Pareto optimal allocation in the Edgeworth box. For each endowment profile on this line, the same PO allocation is an equilibrium at the same prices. Moreover, the Lagrange multipliers are $\omega-$invariant, along the line. In our context, $(p(\overline{\omega}), q(\overline{\omega}))$ is an equilibrium with $\omega-$invariant Lagrange multipliers over the set $E^{-1}(\overline{\omega}, 0, p(\overline{\omega}), q(\overline{\omega}))$.

The motivation for this construction is that, given $\overline{u}$ and fixed resources K, we can parameterize equilibria looking just at the no-trade equilibria and, then, associating the set $E^{-1}(\overline{\omega}, 0, p(\overline{\omega}), q(\overline{\omega}))$ with each $\overline{\omega} \in \Omega^{NT}(K)$.

3. Constrained inefficiency

In the discussion of constrained inefficiency in GEI, the standard approach is to show that, given an equilibrium, generically, there is a profile of portfolios entailing a Pareto improvement. The argument is presented in Geanakoplos and Polemarchakis (1986) and developed in Citanna et al. (1998). It is based on showing that, given the system of eqs.

$$\Xi(\widetilde{p}, \overline{q}, \widetilde{b}) = [\Phi(\widetilde{p}, \overline{q}, \widetilde{b}), (..., u_h(x_h(\widetilde{p}, \overline{q}, \widetilde{b})) - \widehat{u}_h, ...)] = 0,$$

where $\widetilde{b}$ is the portfolio profile, $D_{(\widetilde{p},\widetilde{b})}\Xi(.)$ has, generically, full rank $((S+1)(C-1)+H)$ at each equilibrium. This immediately implies that, for each $\widetilde{a} \in \mathbb{R}^H$, we can find a $\widetilde{b}$, with associated $(\widetilde{b}, \overline{q})-$conditional equilibrium $\widetilde{p}$, such that both $\Phi(\widetilde{p}, \overline{q}, \widetilde{b}) = 0$ and $u_h(x_h(\widetilde{p}, \overline{q}, \widetilde{b})) - \widehat{u}_h = \widetilde{a}$. This is referred to as the submersion approach. Different systems of eqs. $\Phi(.)$ can be selected to describe the equilibrium, but the basic idea is always to add to $\Phi(\widetilde{p}, \overline{q}, \widetilde{b})$ the system of equations $(..., u_h(.) - \widehat{u}_h, ...)$, and to show that the map so obtained has a full rank derivative. If the full rank condition is satisfied, each conceivable Pareto improvement is attainable via an appropriate choice of the portfolios $\widetilde{b}_h$. Clearly, this is a much stronger property than the one required by the definition of CPO. Bear in mind that, for this approach to work, there must be at least H independent policy instruments.[6] Hence, its downside is that it cannot be directly used to study the feasibility of some Pareto improvement in economies with many agents.[7]

In this paper, we do not impose any restriction - but finiteness - on the number of agents and analyze both cases: open sets of economies with non CPO[8] and with CPO equilibria.

[6] This condition is satisfied if $J \geq 2$, as in Geanakoplos and Polemarchakis (1986).
[7] See the Remark on page 89 in Geanakoplos and Polemarchakis (1986).
[8] Since the inefficiency result holds even with just one asset, it can be seen, in a limited way, as a counterexample to a classical claim, originally formulated by Tinberger (1956) and stressed in Citanna et al (1998), according to which to attain H policy objectives, we need a profile of at least H independent policy instruments,

To consider the class of economies with non CPO equilibria, we adopt an approach based on an appropriate optimization problem. This has two advantages: first, for certain purposes, it can be convenient for the interpretation of the results. More relevant, it allows us to obtain some new properties, strengthening the classical results on CP inefficiency in GEI. To establish lack of CPO, it is enough to show that, for some set of economies, at each equilibrium, the necessary conditions for a CPO allocation are violated. Hence, the analysis is purely local. Later on, we will consider a class of economies with a unique CPO equilibrium. There, purely local conditions will not be sufficient, generally speaking, and we will need to take into consideration the entire set of $(b, \overline{q})$ −conditional equilibria.

Let's start formalizing the notion of CPO.

Definition 3. An equilibrium $(\overline{p}, \overline{q})$ is constrained Pareto optimal (CPO) if there is no profile $\widetilde{b} \equiv \left\{..., \widetilde{b}_h, ...\right\}$ with $\sum_h \widetilde{b}_h = 0$ and $\widetilde{p}$ such that $\sum_h \widetilde{z}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h) = 0$, and $u_h(x_h(\widetilde{p}, \overline{q}, \widetilde{b}_h)) \geq u_h\left(x_h(\overline{p}, \overline{q})\right)$, for each h, with at least one strict inequality.

This is the notion adopted in Geanakoplos and Polemarchakis (1986). In our analysis, we allow for $\overline{q}\overline{b}_h \neq \overline{q}\widetilde{b}_h$, for each agent h. To impose the additional constraint $\overline{q}\overline{b}_h = \overline{q}\widetilde{b}_h$, for each h, would have no effect on our results. If an equilibrium is CPO with respect to each portfolio reallocation, it is also CPO when the reallocation is further restricted. The analysis of the set of economies with CP inefficient equilibria goes through with a minor modification of Thm. 2 iv.

It is convenient to split the analysis into two parts. First, we establish that, independently of the number of agents, there are open set of economies with no CPO equilibria. Next, we will establish that, with many agents, there are also open set of economies with CPO equilibria.

3.1. Economies without CPO equilibria

We proceed by adopting an optimization approach. There are several alternative ways to characterize CPO allocations as solutions to an optimization problem. For our purposes, the most convenient is to notice that a CPO price-portfolio profile can be looked at as the optimal solution to the following collection of planning problems: Pick any agent $\overline{h}$. Given an equilibrium $(\overline{p}, \overline{q})$, and a vector $(..., \widehat{V}_h, ...)$, for $h \neq \overline{h}$,

$$\begin{aligned}
\text{choose} \quad (\widetilde{p}, \widetilde{b}) \in &\quad \arg\max \widetilde{V}_{\overline{h}}(\widetilde{p}, \overline{q}, \widetilde{b}_{\overline{h}}) \text{ subject to} \qquad (W_{\overline{h}}) \\
\widehat{V}_h \quad \leq &\quad \widetilde{V}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h), \text{ for each } h \neq \overline{h}, \\
0 \quad = &\quad \widetilde{Z}(\widetilde{p}, \overline{q}, \widetilde{b}) \equiv \sum_h \widetilde{z}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h), \\
0 \quad = &\quad \sum_h \widetilde{b}_h.
\end{aligned}$$

The last two conditions guarantee that $\widetilde{p}$ is an equilibrium conditional on $(\overline{q}, \widetilde{b})$. Assume that $(\overline{p}, \overline{q})$ is an equilibrium. If $(\widetilde{p} = \overline{p}, \widetilde{b} = b(\overline{p}, \overline{q}))$ is not a solution to problem $\left(W_{\overline{h}}\right)$ for some $\overline{h}$, and given $\widehat{V}_h = V_h(\overline{p}, \overline{q})$ for $h \neq \overline{h}$, then the allocation associated with $(\overline{p}, \overline{q})$ is Pareto dominated by the one associated with some other equilibrium conditional on $(\widetilde{b}, \overline{q})$, for some feasible $\widetilde{b}$. Hence, the equilibrium is not CPO. Conversely, if $(\widetilde{p} = \overline{p}, \widetilde{b} = b(\overline{p}, \overline{q}))$ solves the stated optimization problem, for each $\overline{h}$ and $\widehat{V}_h = V_h(\overline{p}, \overline{q})$ for $h \neq \overline{h}$, then, the equilibrium is CPO. We have established the following result:

Lemma 4. An equilibrium price system $(\overline{p}, \overline{q})$ with associated allocation and portfolio profile $\{..., (\overline{x}_h, \overline{b}_h), ...\}$ is CPO if and only if, given $\overline{q}$ and $\widehat{V}_h \equiv V_h(\overline{p}, \overline{q})$, for each h, $(\widetilde{p} = \overline{p}, \widetilde{b} = b(\overline{p}, \overline{q}))$ is an optimal solution to problem $(W_{\overline{h}})$, for each $\overline{h}$.

Given that we are interested in economies with CP inefficient equilibria, we can just focus on a specific agent, $\overline{h} = 1$, and write "problem (W)" instead of $(W_{\overline{h}})$.

The main, possible, advantage in adopting the optimization approach is that the issue of CPO of equilibria may reduce to: under which conditions an equilibrium profile satisfies the FOCs for an optimal solution to (W)? Are they sufficient? In our set-up, there are two difficulties. First, and most obvious, the optimization problem has no nice concavity properties, so that the FOCs of (W) are, generally speaking, not sufficient for an optimal point. This is important when looking for economies with CPO equilibria, but irrelevant here. A more subtle problem is that the FOCs may not even be necessary for a local maximum, when there are many agents. Consider the derivative of the constraints with respect to $(\widetilde{b}_2, ..., \widetilde{b}_H, \widetilde{p})$, replacing $\widetilde{b}_1$ with $\sum_{h>1} \widetilde{b}_h$ and getting rid of the balance constraints for the assets:

$$
\begin{aligned}
& CQ(\widetilde{b}_2, ..., \widetilde{b}_H, \widetilde{p}) \equiv \\
\equiv & \begin{bmatrix} \begin{bmatrix} \nabla_{\widetilde{b}_2} \widetilde{V}_2 - \nabla_{\widetilde{b}_1} \widetilde{V}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nabla_{\widetilde{b}_H} \widetilde{V}_H - \nabla_{\widetilde{b}_1} \widetilde{V}_1 \end{bmatrix} & \begin{bmatrix} \nabla_{\widetilde{p}} \widetilde{V}_2 \\ \vdots \\ \nabla_{\widetilde{p}} \widetilde{V}_H \end{bmatrix} \\ \begin{bmatrix} \left[D_{\widetilde{b}_2} \widetilde{Z} - D_{\widetilde{b}_1} \widetilde{Z} \right] & \cdots & \left[D_{\widetilde{b}_H} \widetilde{Z} - D_{\widetilde{b}_1} \widetilde{Z} \right] \end{bmatrix} & \left[D_{\widetilde{p}} \widetilde{Z} \right] \end{bmatrix} \\
\equiv & \begin{bmatrix} D_{(\widetilde{b}, \widetilde{p})} \widetilde{V}^{\setminus 1} \\ \\ D_{(\widetilde{b}, \widetilde{p})} \widehat{\widetilde{Z}} \end{bmatrix}.
\end{aligned}
$$

Since all the inequality constraints hold as equalities, we need to take into account all of them. At each equilibrium, $\nabla_{\widetilde{b}_h} \widetilde{V}_h = 0$. If $(H-1) \leq (S+1)(C-1)$ and $(H-1)\,J > (S+1)(C-1)$, for (ω, u) in $\mathcal{E}^r$, Thm. 2 iii. and iv. guarantee that the matrix has full row rank, so that the Karush-Kuhn-Tucker (KKT) constraint qualification condition is satisfied. Thus, the FOCs are necessary conditions. Lack of CPO of equilibria follows immediately (Prop. 5).

If $(H-1) > (S+1)(C-1)$, the KKT condition obviously fails. Other, weaker, criteria based on rank invariance of $CQ(.)$[9] are also bound to fail, since rank$CQ(.)$ may increase when we move away from an equilibrium. However, there are open sets of economies such that their regular equilibria may satisfy the Mangasarian - Fromovitz (1967) constraint qualification (MFCQ) criterion, so that the FOCs are indeed necessary for a local maximum. The MFCQ criterion holds at a vector $(\widetilde{b}_2, ..., \widetilde{b}_H, \widetilde{p})$ if $D_{(\widetilde{b}, \widetilde{p})} \widehat{\widetilde{Z}}$ has full rank and there is a vector $\theta \equiv (\theta_b, \theta_p)$ such that $\left[D_{(\widetilde{b}, \widetilde{p})} \widehat{\widetilde{Z}} \right] \theta^T = 0$ and $\left[D_{(\widetilde{b}, \widetilde{p})} \widetilde{V}^{\setminus 1} \right] \theta^T > 0$, where we take into account only the agents with binding constraints (i.e., here, all of them). Our proof of the existence of economies with non CPO equilibria (Prop. 7) is based on constructing, for each no-trade equilibrium, a relatively open subset of economies with regular equilibria satisfying MFCQ, while they cannot satisfy FOCs.

[9] See Janin (1984).

Bear in mind that we are not concerned with existence of CPO prices and allocations. This could be established using a completely different argument, based on continuity and compactness of the relevant constraint set.[10]

Let $\mathcal{L}(\widetilde{p}, \overline{q}, \widetilde{b}; \phi, \mu, \gamma, \eta)$ be the Lagrangian of problem (W). The FOCs are given by

$$\begin{aligned}
D_{\widetilde{p}}\mathcal{L}(.) &= \sum_h \phi_h \left[\nabla_{\widetilde{p}}\widetilde{V}_h(.)\right]^T - \gamma D_{\widetilde{p}}\widetilde{Z}(.) = 0, && (a)\\
D_{\widetilde{b}_h}\mathcal{L}(.) &= \left[..., \phi_h \left[\frac{\partial \widetilde{V}_h(.)}{\partial \widetilde{b}_h^j}\right]^T - \gamma \nabla_{\widetilde{b}_h^j}\widetilde{Z}(.), ...\right] = \eta, \text{ for each } h, && (b)\\
D_{\phi_h}\mathcal{L}(.) &= \widetilde{V}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h) \geq \xi_h, \text{ and}\\
0 &= \left[\widetilde{V}_h(\widetilde{p}, \overline{q}, \widetilde{b}_h) - \xi_h\right]\phi_h, \text{ for each } h > 1, \ \phi_h \geq 0, && (c)\\
D_{\gamma}\mathcal{L}(.) &= -\widetilde{Z}(\widetilde{p}, \overline{q}, \widetilde{b}) = 0, && (d)\\
D_{\eta}\mathcal{L}(.) &= -\sum_h \widetilde{b}_h = 0, && (f)
\end{aligned}$$

where, with some abuse of notation, we use $\phi_1 \equiv 1$. $\{\phi, \mu, \gamma, \eta\}$ are the vectors of Lagrange multipliers.

We need to distinguish two cases. The first is basically the one considered in Geanakoplos and Polemarchakis (1986): the number of agents is smaller than the one of the, non-numeraire, commodity prices: $H \leq (S+1)(C-1)$. If $(H-1)J \geq (S+1)(C-1)$, generically, at each equilibrium KKT constraint qualification holds and the FOCs of problem (W) do not. Hence, equilibria are not CPO. Our result is not encompassed by the one already established in the literature, since our lower bound on the number of agents may be smaller than the one in Geanakoplos and Polemarchakis (1986), $H \geq 2(C-1)$. However, the substantive difference is really tiny. The result is presented here mostly for completeness.[11]

Proposition 5. Under the maintained assumptions, if $H \leq (S+1)(C-1)$ and $(H-1)J \geq (S+1)(C-1)$, each equilibrium allocation is not CPO for economies in the generic set $\mathcal{E}^r$.

Proof. By Thm. 2 iii. and iv., the matrix $CQ(\widetilde{b}_2, ..., \widetilde{b}_H, \widetilde{p})$ defined above has full row rank at each equilibrium. Thus, the KKT constraint qualification criterion is satisfied and the FOCs are necessary for a local maximum of (W). At each equilibrium, $\frac{\partial \widetilde{V}_h(.)}{\partial \widetilde{b}_h^j} = 0$. The system of FOCs essentially reduces to

$$\begin{aligned}
D_{\widetilde{p}}\mathcal{L}(.) &= \sum_h \phi_h \left[\nabla_{\widetilde{p}}\widetilde{V}_h(.)\right]^T - \gamma D_{\widetilde{p}}\widetilde{Z}(.) = 0, && (a)\\
D_{\widetilde{b}_h}\mathcal{L}(.) &= -\gamma\left[..., \left[D_{\widetilde{b}_h}\widetilde{Z}(.) - D_{\widetilde{b}_1}\widetilde{Z}(.)\right]^T, ...\right] = 0, \text{ for each } h > 1. && (b')
\end{aligned}$$

By Thm. 2 iv., and since $(H-1)J \geq (S+1)(C-1)$, (b') implies $\gamma = 0$. Hence, by Thm. 2 iii., $\sum_h \phi_h \left[\nabla_{\widetilde{p}}\widetilde{V}_h(.)\right]^T = 0$ if and only if $\phi = 0$. This contradicts the fact that $\phi_1 = 1$. Hence, the system of - necessary - FOCs has no solution at each equilibrium of $(\omega, u) \in \mathcal{E}^r$. Therefore, equilibria are not CPO. ■

[10] Werner (1991) studies CPO allocations in GEI. Their possible non-existence is due to changes in the rank of the payoff matrix. This issue cannot arise with numeraire assets.

[11] This is also why it is not worthwhile to investigate if our approach, with some adjustment, works, as it probably does, with a less tight lower bound on H, limiting ourselves to the easy case.

Let's now turn to the most interesting case: CP inefficiency when $H > (S+1)(C-1)$. Later on, we will show that there are open sets of economies with a unique, CPO equilibrium. Hence, CP inefficiency cannot be a generic properties, when there are many agents. However, we can still ask: first, how common are economies with non CPO equilibria? Second: can we provide simple conditions guaranteeing lack of CPO? The answer to the first is: quite common. In Prop. 7, we show that, given any no-trade equilibrium satisfying condition iv. of Thm. 2, there is a open set of economies whose equilibrium is not CPO. We provide a (limited) answer to the second: lack of CPO typically holds if, for each agent, the present value of the vector of their net trades in good $s1$, $s = 0, ...S$, computed according to his/her own risk-neutral probabilities, is strictly positive. Clearly, it would be nice to have a result just in terms of observable (prices, consumption, or net trades, and portfolios). This is, however, impossible: the normalized vector of Lagrange multipliers, i.e., the personalized risk-neutral probabilities, necessarily play a key role.

As a preliminary step, we show that, in each open neighborhood of an economy with a no-trade equilibrium, there are economies with a regular equilibrium satisfying MFCQ.

Lemma 6. Let $H > (S+1)(C-1)$. Given $\overline{u}$, consider any $\overline{\omega} \in \Omega^{NT}(K)$ and any open set $B(\overline{\omega}, \overline{u})$. Then, there is some $(\omega, u) \in B(\overline{\omega}, \overline{u})$ with a regular equilibrium satisfying MFCQ.

The proof is in Appendix. Bear in mind that $H > (S+1)(C-1)$ implies that $(H-1)J \geq (S+1)(C-1)$.

Proposition 7. Let $H > (S+1)(C-1)$. Then, for each no-trade, Pareto inefficient equilibrium $(\overline{\omega}, 0), (p(\overline{\omega}), q(\overline{\omega}))$, there is a relatively open subset of $E^{-1}((\overline{\omega}, 0), (p(\overline{\omega}), q(\overline{\omega})))$ such that, for each economy in this subset, the associated equilibrium allocation is not CPO.

Proof. Consider any Pareto inefficient, no-trade equilibrium $(\overline{\omega}, 0), (p(\overline{\omega}), q(\overline{\omega}))$. By Lemma 6, we can pick $\omega \in E^{-1}((\overline{\omega}, 0), (p(\overline{\omega}), q(\overline{\omega})))$ such that MFCQ holds, so that the FOCs of (W) are necessary for an optimal solution. The construction in Lemma 6 implies that both $\left[D_{\widetilde{p}}\widetilde{V}^{\setminus 1}\right]\overrightarrow{dp} >> 0$ and $\left[D_{\widetilde{p}}\widetilde{V}_1\right]\overrightarrow{dp} > 0$. Hence, by Stiemke thm. of the alternatives there is no vector $\phi \geq 0$ such that $\sum_h \phi_h \left[D_{\widetilde{p}}\widetilde{V}_h\right]^T = 0$. This implies that the equilibrium cannot satisfy the FOCs. Given that there is no loss of generality in assuming that the equilibrium is regular, there is a relatively open subset of $E^{-1}((\overline{\omega}, 0), (p(\overline{\omega}), q(\overline{\omega})))$ such that the associated economies have a non CPO equilibrium. ■

In our construction, we have established the existence of relatively open subsets of $E^{-1}((\overline{\omega}, 0), (p(\overline{\omega}), q(\overline{\omega})))$ with a non CPO equilibrium. However, in principle, economies in this set could have some other, possibly CPO, equilibria. We can rule out this possibility by restricting ourselves to economies with a unique equilibrium.

Corollary 8. Let $H > (S+1)(C-1)$. Then, there is an open subsets of $\mathcal{E}$, $\mathcal{E}^{NCPO}$, such that, for each $(\omega, u) \in \mathcal{E}^{NCPO}$, the associated unique equilibrium allocation is CP inefficient.

Proof. Uniqueness and regularity of the equilibrium for each $(\overline{\omega}, \overline{u})$ with a PO initial endowment imply that there is some open ball $B(\overline{\omega}, \overline{u})$ such that, for each $(\omega, u) \in B(\overline{\omega}, \overline{u})$, the equilibrium is unique and regular. Pick any $(\omega, u) \in B(\overline{\omega}, \overline{u})$ with a Pareto inefficient no-trade equilibrium and apply Prop. 7. By construction, all the economies have a unique, regular, CP inefficient equilibrium. This property can be immediately extended to all (ω, u) in some open neighborhood. ■

Remark 9. In a neighborhood of each no-trade, Pareto inefficient equilibrium, we can construct open subsets of economies such that MFCQ holds and FOCs are violated at equilibrium, as in the proof of Prop. 7. We could also construct open sets of economies such that MFCQ and FOCs are satisfied at the equilibrium. Equilibria of economies in this set could be CPO. In fact, as shown in Prop. 12, they are CPO when the initial no-trade equilibrium is chosen appropriately. The key issue for CPO is the row span of the matrix $\Lambda(\lambda, z)$. At each no-trade equilibrium, $\Lambda(\lambda, z)$ is trivial. Arbitrarily small perturbations of the endowments allow us to associate with the same equilibrium $(\overline{p}, \overline{q})$ matrices $\Lambda(\lambda, z)$ spanning different subspaces of $\mathbb{R}^H$. For some of them, $\text{span}\Lambda(\lambda, z)^T \cap \mathbb{R}^H_+ \neq \emptyset$, so that the equilibrium is definitely not CPO. For others, $\text{span}\Lambda(\lambda, z)^T \cap \mathbb{R}^H_+ = \emptyset$, so that the equilibrium could be CPO. In Prop. 12, we will show that they are actually CPO when preferences are sufficiently close to be identical and homothetic.

A simple sufficient condition for CP suboptimality is provided in the following Proposition.

Proposition 10. Under the maintained assumptions, if $H > (S+1)(C-1)$, for each $(\omega, u) \in \mathcal{E}^r$, each equilibrium allocation such that, for each h,

$$\sum_s \lambda_h^s \left(\overline{p}(\omega, u), \overline{q}(\omega, u)\right) z_h^{s1} \left(\overline{p}(\omega, u), \overline{q}(\omega, u)\right) > 0,$$

is CP inefficient.

Proof. Restrict the analysis to $(\omega, u) \in \mathcal{E}^r$. The proof is essentially identical to the one of Prop. 7. Observe that, for each s,

$$\sum_{c>1} \overline{p}^{sc} z_h^{sc} (\overline{p}, \overline{q}) = r^s b_h \left((\overline{p}, \overline{q})\right) - z_h^{s1} (\overline{p}, \overline{q}), \text{ and } \sum_{c>1} \overline{p}^{0c} z_h^{0c} (\overline{p}, \overline{q}) = -\overline{q} b_h (\overline{p}, \overline{q}) - z_h^{s1} (\overline{p}, \overline{q})$$

and that, by the noarbitrage conditions, for each h:

$$\begin{aligned} -\sum_s \lambda_h^s (\overline{p}, \overline{q}) \sum_{c>1} \overline{p}^{sc} z_h^{sc} (\overline{p}, \overline{q}) &= -\left[..., \lambda_h^s (\overline{p}, \overline{q}), ...\right] Y(\overline{q}) b_h (\overline{p}, \overline{q}) \\ +\sum_s \lambda_h^s (\overline{p}, \overline{q}) z_h^{s1} (\overline{p}, \overline{q}) &= \sum_s \lambda_h^s (\overline{p}, \overline{q}) z_h^{s1} (\overline{p}, \overline{q}). \end{aligned}$$

Hence, set $\overrightarrow{dp} = \left[\overline{p}^{02}, ..., \overline{p}^{SC}\right] \in \mathbb{R}^{(S+1)(C-1)}_{++}$:

$$\overrightarrow{dp} \left[\nabla_{\overline{p}} \widetilde{V}\right]^T = \left[..., \sum_s \lambda_h^s (\overline{p}, \overline{q}) z_h^{s1} (\overline{p}, \overline{q}), ...\right],$$

and, by assumption, this is a strictly positive vector. Hence, there is no $\phi \in \mathbb{R}^H_+$ such that $\sum_h \phi_h \left[\nabla_{\overline{p}} \widetilde{V}_h\right] = 0$. Therefore, the equilibrium allocation does not satisfy the FOCs, and it must be constrained inefficient. ■

Evidently, the previous results are non-generic, when there are many agents. Still, they are of some interest for at least three reasons:

1. they may hold even if there is just one asset. Thus, we can obtain a Pareto improvement when the number of independent policy instruments is smaller than the number of agents,
2. they hold for each sufficiently large, but finite, number of agents, without requiring the

use of additional policy instruments, such as period 0 lump-sum taxes, as in Citanna et al. (1998),

3. they illustrate how the same equilibrium allocation may, or may not, be CPO depending upon the distribution of the initial endowments.

We conclude this section presenting a parametric example. Since $H > (S+1)(C-1)$, Geanakoplos and Polemarchakis (1986) does not apply. However, there is a range of values of the parameters of the utility functions such that the condition stated in Prop. 10 is satisfied, so that the equilibrium allocation is not CPO.

Example 11. Consider an economy with four agents, three spots, two goods in each spot and one asset, inside money. Preferences are Cobb-Douglas,

$$\begin{aligned} u_h(x_h) &= \alpha_h^0 \ln x_h^{01} + \left(1-\alpha_h^0\right) \ln x_h^{02} + \beta_h \left(\alpha_h^1 \ln x_h^{11} + \left(1-\alpha_h^1\right) \ln x_h^{12}\right) \\ & \quad + \gamma_h \left(\alpha_h^2 \ln x_h^{21} + \left(1-\alpha_h^2\right) \ln x_h^{22}\right), \end{aligned}$$

with the following values of the parameters

$$\begin{bmatrix} & a_h^{01} & a_h^{11} & a_h^{21} & \beta_h & & \gamma_h \\ h=1 & 0.05 & 0.15 & \frac{7}{12} & \left(\frac{2}{3}-\frac{5}{3}\gamma_1\right) & .3 & \left(0.259\,52, \frac{2}{5}\right) \\ h=2 & 0.95 & 0.90 & 0.05 & \left(\frac{3}{7}-\frac{3}{5}\gamma_2\right) & .1 & (0, 0.377\,2) \\ h=3 & 0.95 & 0.90 & 0.8 & \frac{7}{5}(1-\gamma_3) & 0.5 & (0,1) \\ h=4 & 0.05 & 0.05 & \frac{4}{7} & \left(1-\frac{5}{7}\gamma_4\right) & 1.3 & (1.095\,7, 1.4) \end{bmatrix}.$$

The endowment vectors are $\omega_1 = [1,7,2,2,0,2]$, $\omega_2 = [7,1,1,1,3,1]$, $\omega_3 = [2,2,4,4,6,0]$, and $\omega_4 = [2,2,3,3,1,7]$. A straightforward computation shows that there is an equilibrium, $(\overline{p}, \overline{q}) = (1,1,1,1,1)$, with associated consumption, excess demand and Lagrange multipliers described below

$$\begin{bmatrix} & z_h^{01} & z_h^{11} & z_h^{21} & x_h^{01} & x_h^{11} & x_h^{21} & \overline{\lambda}_h^0 & \overline{\lambda}_h^1 & \overline{\lambda}_h^2 \\ h=1 & -0.65 & -1.25 & \frac{7}{4} & 0.35 & 0.75 & \frac{7}{4} & \frac{1}{7} & \left(\frac{2}{3}-\frac{5}{3}\gamma_1\right)\frac{1}{5} & \frac{1}{3}\gamma_1 \\ h=2 & 0.65 & 1.70 & -2.75 & 6.65 & 2.7 & 0.25 & \frac{1}{7} & \left(\frac{3}{7}-\frac{3}{5}\gamma_2\right)\frac{1}{3} & \frac{1}{5}\gamma_2 \\ h=3 & 2.75 & 2.30 & -2 & 4.75 & 6.30 & 4 & \frac{1}{5} & \frac{7}{5}(1-\gamma_3)\frac{1}{7} & \frac{1}{5}\gamma_3 \\ h=4 & -1.75 & -2.75 & 3 & 0.25 & 0.25 & 4 & \frac{1}{5} & \left(1-\frac{5}{7}\gamma_4\right)\frac{1}{5} & \frac{1}{7}\gamma_4 \end{bmatrix}.$$

By direct computation, we obtain

$$\sum_s \overline{\lambda}_h^s \overline{z}_h^{s1} = [1.0\gamma_1 - 0.259\,52, 0.335\,71 - 0.89\gamma_2, 1.01 - 0.86\gamma_3, 0.82143\gamma_4 - 0.9],$$

which can be made strictly positive for an appropriate choice of the coefficients $(\gamma_1, ..., \gamma_4)$. Figure 1 reports the utility gains associated with a portfolio reallocation

$$\overrightarrow{db} = [-80.671, 6.792\,5, -3.125\,9, 77.\,004]\, d, \;\; d > 0.$$

Evidently, there is a Pareto improving feasible direction of portfolio reallocation.

3.2. Economies with CPO equilibria

Our last result is that there are also open sets of economies with CPO equilibria. The argument is more elaborate, because we need to compare each equilibrium with the entire set of

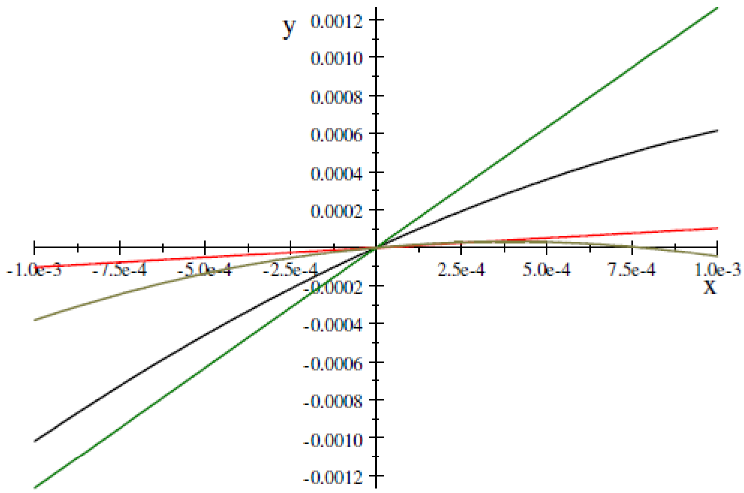

Figure 1: Utility gains

$\widetilde{b}-$conditional equilibria. Prop. 12 shows that equilibria are CPO for some set of economies with equilibrium allocations close to a PO one, and preferences close to be homothetic and identical across agents. This means that lack of CPO cannot be a generic property of equilibria, when there are many agents. An heuristic argument to understand this property is based on the optimization approach pursued above. If constraint qualification holds, the essential issue for CPO, or lack of it, is if there is a strictly positive solution to the system of eqs. $\sum_h \phi_h \left[\nabla_{\widetilde{p}} \widetilde{V}_h(.)\right]^T = 0$. Given the structure of $\nabla_{\widetilde{p}} \widetilde{V}_h(.)$ and market clearing, when the equilibrium allocation is PO there is always a solution $\overline{\phi} >> 0$. If $H \leq (S+1)(C-1)$, the existence of a non-trivial solution is not robust to perturbations of the economy. Thus, lack of CPO may be a generic property, as shown in Prop. 5. Instead, its existence is robust if $H > (S+1)(C-1)$, so that there is a strictly positive solution $\overline{\phi}$, for some open set of economies in any neighborhood of a Pareto optimal allocation. This suggests that, at least locally, these equilibria may be CPO, since they satisfy the necessary FOCs. The next proposition shows that this is actually the case.

Proposition 12. Under the maintained assumptions, if $H > (S+1)(C-1)$, the set

$$\mathcal{E}^{CPO} \equiv \left\{(\omega, u) \in \mathcal{E}^{CPO} | \text{all the equilibria are } \mathcal{CPO}\right\}$$

has non-empty interior.

The proof is in Appendix. Here, we just provide its outline. We start with an economy $(\overline{\omega}, \overline{u})$ with identical, homothetic preferences and a PO endowment. The properties of the utility functions guarantee that each economy has a unique equilibrium price, which does not depend upon the portfolio allocation. Hence, the second order effects of price changes on the equilibrium level of utility are negligible. In fact, the indirect utility functions, evaluated taking into account the price adjustments, are strictly concave in $\widetilde{b}$. By continuity, the same properties hold for economies sufficiently close to $(\overline{\omega}, \overline{u})$. The proof that the equilibria of these economies are CPO rests crucially upon the fact that we are dealing with open sets of economies contained in some small neighborhood of a Pareto optimal allocation.

To avoid misunderstandings, it may be convenient to recall here Remark 9 above. Consider economies with a unique CPO equilibrium and the associated no-trade economy, say $(\widehat{\omega}, \widehat{u})$. Assume that preferences are sufficiently close to be identical across agents and homothetic.

Then, provided that Thm. 2 iv. holds at $(\widehat{\omega}, \widehat{u})$, $E^{-1}((\widehat{\omega}, 0), p(\widehat{\omega}), q(\widehat{\omega}))$ contains both economies $(\omega', \widehat{u})$ with a unique CPO equilibrium and economies $(\omega", \widehat{u})$ with a unique, CP inefficient equilibrium.

4. Concluding remarks

In this paper, we have shown that, in economies with a large, but finite, number of agents, typically there are open sets of economies with CPO equilibria, and other open sets of economies with CP inefficient equilibria. We have also provided a simple sufficient condition for lack of CPO for economies in a generic set.

To use problem (W) as a building block for the analysis of CP inefficiency is of some interest, because it allows us to make transparent the nature of the efficiency problem of GEI equilibria. With complete markets, or in the non-generic cases where equilibria in GEI are always CPO, pecuniary externalities induced by portfolio reallocations can be aggregated over the set of agents using positive weights in such a way that they cancel out. Hence, any gain in utility for some agent must be compensated by a loss in utility for some other agent. Equilibria are CPO. In GEI economies, this happens, for instance, when there cannot be pecuniary externalities (as in the case of identical homothetic preferences) or when Lagrange multipliers are collinear, so that market clearing immediately implies that the effects of these externalities must disappear in the aggregate. Apart from non-generic cases of this sort, portfolio-induced pecuniary externalities may lead to a Pareto improvement. This is the key result of Prop. 7 and of its corollary. On the other hand, given preferences and any asset structure, we can always find a Pareto optimal initial endowment profile. The associated equilibrium is, obviously, CPO. Prop. 12 has shown that in each open neighborhoods of these exceptional endowment profiles there are open sets of economies with a unique, CPO equilibrium, when preferences are sufficiently close to be identical and homothetic. Both results are robust to utility perturbations. Therefore, there are no robust classes of preferences such that CPO (or lack of CPO) holds generically, with many agents. From this viewpoint, we believe that our results settle most of the open issues concerning the general CPO properties of equilibria in GEI. A remaining one is if there are more appealing sufficient conditions guaranteeing CPO of equilibria, or its lack of, for some restricted class of economies.

5. Appendix

Proof of Theorem 2. i. is standard, see Geanakoplos and Polemarchakis (1986). Let $\mathcal{E}^R$ be the generic set of economies such that i. holds.
ii. Since R is in general position, we just need to consider square submatrices $\left[-q, \underline{R}^T\right]$, where $\underline{R}$ is given by a collection of $(J-1)$ rows of R, without loss of generality, rows $1, ..., J-1$. Define the map $\Xi(\psi, \gamma) \equiv \left[-\left[R^T\psi\right]\gamma^0 + \underline{R}^T[\gamma^1, ..., \gamma^{J-1}]\right]$. We can take $\Xi : \overset{\circ}{\Delta}^{S-1} \times S^{J-1} \to \mathbb{R}^J$, where S^{J-1} is the unit sphere in $\mathbb{R}^J$, while $\overset{\circ}{\Delta}^{S-1}$ is some compact manifold without boundary contained in the unit simplex. Since R is in general position, and $\gamma \neq 0$, $\Xi(\psi, \gamma) = 0$ implies that $\gamma^0 \neq 0$. Given that $D_{(\psi^1, ..., \psi^S)}\Xi(\psi, \gamma) = -R^T\gamma^0$, $\Xi \pitchfork 0$. Therefore, there is an open, dense subset of $\overset{\circ}{\Delta}^{S-1}$ such that $\Xi_\psi \pitchfork 0$. Then, $\Xi_\psi^{-1}(0) = \emptyset$, because $\Xi_\psi : S^{J-1} \to \mathbb{R}^J$. Iterating the procedure for all possible collections of (J-1) rows of R, and taking intersections, we obtain that, for a generic choice of the vector ψ of Arrow state-prices, $Y(\psi R)$ is in general position.

Restrict the analysis to $(\omega, u) \in \mathcal{E}^R$ and, without loss of generality, assume that, for each h, the equilibrium allocation is different at each one of the distinct equilibria of $(\overline{\omega}, \overline{u})$. Given any $(\overline{\omega}, \overline{u})$, pick any equilibrium $(\overline{p}, \overline{q})$ such that $Y(\overline{q})$ is not in general position and $\overline{q} = \overline{\psi} R$. As we have seen, an arbitrarily small perturbation of $\overline{\psi}$ suffices to guarantee that, at the new asset prices $q' \equiv \psi' R$, $Y(q')$ is in general position. Evidently, for ψ' close to $\overline{\psi}$, for each h we can find a vector λ'_h close to the Lagrange multiplier at the equilibrium of the initial economy, $\overline{\lambda}_h$, and such that $\lambda'_h Y(q') = 0$. For each h, replace the equilibrium consumption bundle $\overline{x}_h$ with the bundle x'_h, defined as follows: $x_h^{sc\prime} = \overline{x}_h^{sc}$ for each $sc \neq 01$, $x_h^{01\prime} = \overline{x}_h^{01} + (\overline{q} - q')\overline{b}_h$. Evidently, x'_h is budget feasible at prices $(\overline{p}, q')$. Now, consider a locally linear perturbation of the utility function, obtained replacing $\overline{u}_h(.)$ with

$$u'_h(x_h) \equiv \overline{u}_h(x_h) + \theta_\varepsilon(x_h) \sum_s \left(\lambda_h^{s\prime} \overline{p}^s - \frac{\partial \overline{u}_h(x_h)}{\partial x_h}|_{x_h = x'_h} \right) x_h^s,$$

where $\theta_\varepsilon(x_h)$ is a smooth "bump" function taking the value 1 on the open ball of radius ε centered on $\overline{x}_h \equiv x_h(\overline{p}, \overline{q}; \overline{\omega}_h, \overline{u}_h)$, $B_\varepsilon(\overline{x}_h)$, the value 0 at $x_h \notin B_{2\varepsilon}(\overline{x}_h)$. It is easy to check that, for the new economy, $(\overline{p}, q')$ is an equilibrium with allocation and portfolio profile $\left\{ ..., \left(x'_h, \overline{b}_h\right), ... \right\}$. Choosing ε sufficiently small, we can guarantee that, given any pair of equilibria of the initial economy $(\overline{\omega}, \overline{u})$, $(\overline{p}, \overline{q})$ and $(\widehat{p}, \widehat{q})$, $B_{4\varepsilon}(x_h(\overline{p}, \overline{q})) \cap B_{4\varepsilon}(x_h(\widehat{p}, \widehat{q})) = \emptyset$, so that we can locally perturb $\overline{u}_h$ in different directions at the distinct equilibria. Given that the number of equilibria is finite, by iterating the procedure, given any open neighborhood of $(\overline{\omega}, \overline{u})$, we can construct a profile $\left(\overline{\omega}, u''\right)$ contained in the neighborhood and such that, at each equilibrium, $Y(q')$ is in general position. Given that, for q' sufficiently close to $\overline{q}$, $u'_h(x_h)$ can be made arbitrarily close to $\overline{u}_h(x_h)$, this establishes density of the set $(\omega, u) \in \mathcal{E}^R$ satisfying ii. Its openness follows immediately from regularity of equilibria for $(\omega, u) \in \mathcal{E}^R$. Let $\mathcal{E}^{gp}$ be the generic set of economies such that i. and ii. hold.

iii. This also follows by an iterated application of the transversality thm. Thus, we just outline the proof. First, observe that,[12]

(a) Generically, at each equilibrium $(\overline{p}, \overline{q})$, for each $\overline{h}$, $\frac{\lambda_h^s(\overline{p}, \overline{q})}{\lambda_h^\sigma(\overline{p}, \overline{q})} \neq \frac{\lambda_{\overline{h}}^s(\overline{p}, \overline{q})}{\lambda_{\overline{h}}^\sigma(\overline{p}, \overline{q})}$ for each $s \neq \sigma$, for some h. This can be established exploiting the same - locally - linear utility perturbation described in ii.,

(b) This implies that, generically, $\sum_h \frac{\lambda_h^s(\overline{p}, \overline{q})}{\lambda_h^\sigma(\overline{p}, \overline{q})} z_h^s(\overline{p}, \overline{q}) \neq 0$, for each $s \neq \sigma$, and each σ,at each equilibrium $(\overline{p}, \overline{q})$,

(c) Generically, for each s, $\begin{bmatrix} \cdots & [\lambda_h^s(\overline{p}, \overline{q})\, z_h^s(\overline{p}, \overline{q})]^T & \cdots \end{bmatrix}$ has full row rank $(C-1)$ at each equilibrium $(\overline{p}, \overline{q})$. This follows immediately from the fact that, generically, at each equilibrium and for each s, $\begin{bmatrix} \cdots & [z_h^s(\overline{p}, \overline{q})]^T & \cdots \end{bmatrix}$ has full row rank.

Restrict the analysis to the set of economies such that (a, b, c) hold and look at the two possible cases:

(I) $(S+1)(C-1) \geq H$. Consider the system of eqs.

$$\Lambda(\lambda, z)\phi^T = 0, \ \phi \in S^{H-1}.$$

$D_\omega \left[\Lambda(\lambda, z)\phi^T\right]$ spans the directional derivative

$$\begin{bmatrix} Diag\left(\phi_1 \lambda_1 - \phi_{\overline{h}} \lambda_{\overline{h}}\right) & \cdots & Diag\left(\phi_H \lambda_H - \phi_{\overline{h}} \lambda_{\overline{h}}\right) \end{bmatrix},$$

for each $\overline{h}$. If there is $\overline{h}$ such that $\phi_{\overline{h}} = 0$, $D_\omega \left[\Lambda(\lambda, z)\phi^T\right]$ has full rank $(S+1)(C-1)$. Otherwise, $\phi_h \neq 0$, for each h. If rank$D_\omega \left[\Lambda(\lambda, z)\phi^T\right] < (S+1)(C-1)$, it must be that, for some $s = \sigma$,

[12] (a) and (b) are also exploited as properties (2.2.) and (2.3) in Citanna et al. (1998). Here, we impose that they hold for all pairs s, s', instead than just for $s = 0, 1$.

$\phi_h\lambda_h^\sigma - \phi_{\overline{h}}\lambda_{\overline{h}}^\sigma = 0$, for each h, i.e., $\frac{\phi_h}{\phi_{\overline{h}}} = \frac{\lambda_{\overline{h}}^\sigma}{\lambda_h^\sigma}$. However, $\Lambda(\lambda, z)\phi = 0$ and $\frac{\phi_h}{\phi_{\overline{h}}} = \frac{\lambda_{\overline{h}}^\sigma}{\lambda_h^\sigma}$ is impossible, because it violates (b).

(II) $\quad (S+1)(C-1) < H$. Consider the system of eqs.

$$\Lambda(\lambda, z)^T\varphi = 0,\ \varphi \in S^{(S+1)(C-1)-1}.$$

$D_\omega\left[\Lambda(\lambda, z)^T\varphi^T\right]$ spans the directional derivative

$$\begin{bmatrix} \cdots & \varphi^{sc} \begin{bmatrix} \lambda_1^s & & \\ & \ddots & \\ & & \lambda_{H-1}^s \\ -\lambda_H^s & & -\lambda_H^s \end{bmatrix} & \cdots \end{bmatrix}.$$

Suppose that, for at least two spots s and s', there exists $c(s)$ and $c(s')$ such that $\varphi^{sc(s)} \neq 0$ and $\varphi^{s'c(s')} \neq 0$. Then, it must be $\alpha_h\lambda_h^s = \lambda_H^s$ and $\alpha_h\lambda_h^{s'} = \lambda_H^{s'}$, for each h, which violates (a). On the other hand, $\varphi^{s'} = 0$ for each $s' \neq s$, for some s, is impossible in view of (c). It follows that $\Lambda(\lambda, z)$ has maximum rank at each equilibrium for economies in some open, dense subset of $\mathcal{E}^{gp}$.

iv. Restrict the analysis to $(\omega, u) \in \mathcal{E}^{gp}$. At $\widetilde{p} = \overline{p}$,

$$\left[\frac{\partial\widetilde{z}_h^{sc}(\widetilde{p}, \overline{q}, \widetilde{b}_h)}{\partial\widetilde{b}_h^j} - \frac{\partial\widetilde{z}_1^{sc}(\widetilde{p}, \overline{q}, \widetilde{b}_1)}{\partial\widetilde{b}_1^j}\right] \equiv \left[\frac{\partial z_h^{sc}(\overline{p}, \overline{q})}{\partial m_h^s} - \frac{\partial z_1^{sc}(\overline{p}, \overline{q})}{\partial m_1^s}\right] r^{sj}$$

where m_h^s is h's income in state s.

Define the $((S+1)(C-1) \times (H-1)J)$ $-$dimensional matrix

$$G(\overline{p}, \overline{q}; \omega, u) \equiv \begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots & \left[\frac{\partial z_h^s(\overline{p}, \overline{q})}{\partial m_h^s} - \frac{\partial z_1^s(\overline{p}, \overline{q})}{\partial m_1^s}\right]^T r^{sj} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix},$$

and the system of equations

$$\Theta(\overline{p}, \overline{q}, \alpha; \omega, u) = \begin{bmatrix} \Phi(\overline{p}, \overline{q}; \omega, u) \\ \\ \left[G(\overline{p}, \overline{q}; \omega, u)\right]^T \alpha^T \end{bmatrix} = 0$$

with $\alpha \in S^{(S+1)(C-1)-1}$, the unit sphere in $\mathbb{R}^{(S+1)(C-1)}$. Under standard technical conditions, by the transversality thm., if $\Theta(\overline{p}, \overline{q}, \alpha; \omega, u) \pitchfork 0$, there exists an open, dense subset of $\mathcal{E}^{gp}$, $\mathcal{E}^r$, such that, for each $(\omega, u) \in \mathcal{E}^r$, $\Theta_{(\omega,u)}(\overline{p}, \overline{q}, \alpha) \pitchfork 0$. Since $\Theta_{(\omega,u)}(.)$ maps $\mathbb{R}^{(S+1)(C-1)+J} \times S^{(S+1)(C-1)-1}$ into $\mathbb{R}^{(S+1)(C-1)+J} \times \mathbb{R}^{(H-1)J}$ and $(H-1)J > (S+1)(C-1)-1$, $\Theta_{(\omega,u)}(\overline{p}, \overline{q}, \alpha) \pitchfork 0$ implies that $\Theta_{(\omega,u)}^{-1}(0) = \emptyset$, i.e., that, at each equilibrium, the matrix $G_{(\omega,u)}(\overline{p}, \overline{q})$ has full row rank $(S+1)(C-1)$. Let's show that $\Theta(\overline{p}, \overline{q}, \alpha; \omega, u) \pitchfork 0$. Consider

$$D_{(\omega,u)}\Theta(\overline{p}, \overline{q}, \alpha; \omega, u) = \begin{bmatrix} D_\omega\Phi(\overline{p}, \overline{q}; \omega, u) & D_{\overrightarrow{du}}\Phi(\overline{p}, \overline{q}; \omega, u) \\ \\ * & D_{\overrightarrow{du}}\left[\left[G(\overline{p}, \overline{q}; \omega, u)\right]^T \alpha^T\right] \end{bmatrix}.$$

It is straightforward to show that $D_\omega\Phi(\overline{p}, \overline{q}; \omega, u)$ has full rank $((S+1)(C-1)+J)$. We will consider perturbations $\overrightarrow{du}$ of the utility functions which do not affect $\Phi(\overline{p}, \overline{q}; \omega, u)$, so that $D_{\overrightarrow{du}}\Phi(\overline{p}, \overline{q}; \omega, u) = 0$, while they change by 1 the derivatives $\frac{\partial z_h^{sc}}{\partial m_h^s}$ (and, accordingly, $\frac{\partial z_h^{s1}}{\partial m_h^s}$), for

$h = 2, ..., H$. Perturbations with these properties exist (see Geanakoplos and Polemarchakis (1980)).

We need to consider two different cases:

a. there are at least J distinct states such that, for some $c(s)$, $\alpha^{sc(s)} \neq 0$. Given ii. above, we can assume, without loss of generality, that $s = 1, ..., J$. Define $\overline{\alpha} \equiv \left[\alpha^{1c(1)}, ..., \alpha^{Jc(J)}\right]$, a J dimensional vector with no zero coordinate. Perturb, as described above, the utility functions of each agent $h > 1$, changing by 1 the derivatives $\frac{\partial z_h^{sc(s)}}{\partial m_h^s}$. Then,

$$D_{\overrightarrow{du}}\left[\left[G(\overline{p}, \overline{q}; \omega, u)\right]^T \alpha\right] = \begin{bmatrix} \ddots & & \\ & \left[\left[r^1\right]^T \alpha^{1c(1)} \quad \cdots \quad \left[r^J\right]^T \alpha^{Jc(J)} \right] & \\ & & \ddots \end{bmatrix},$$

a block diagonal matrix. Given that $(\omega, u) \in \mathcal{E}^{gp}$, $Y(q)$ is in general position. Hence, the nontrivial elements are given by a full rank matrix. Evidently, $D_{\overrightarrow{du}}\left[\left[G(.)\right]^T \alpha^T\right]$ has maximal rank $(H-1)J$.

b. there are at most $(J-1)$ states, without loss of generality, as above, $s = 1, ..., J-1$, such that, for some $c(s)$, $\alpha^{sc(s)} \neq 0$. We now show that this is impossible, for a generic set of economies. By contradiction, assume that there is $\widetilde{\alpha}$ such that $\left[G(.)\right]^T \widetilde{\alpha}^T = 0$ and $\widetilde{\alpha}^{sc} \neq 0$ for some c in less than J states. We can explicitly write

$$\begin{aligned} &\left[G(.)\right]^T \widetilde{\alpha} \\ = &\begin{bmatrix} \vdots \\ -\left[\sum_{c>1}\left(\frac{\partial z_h^{0c}}{\partial m_h^0} - \frac{\partial z_1^{0c}}{\partial m_1^0}\right)\widetilde{\alpha}^{0c}\right]\overline{q}^T + \sum_{s>0}\left[\sum_{c>1}\left(\frac{\partial z_h^{sc}}{\partial m_h^s} - \frac{\partial z_1^{sc}}{\partial m_1^s}\right)\widetilde{\alpha}^{sc}\right] r^{sT} \\ \vdots \end{bmatrix} \\ = &\begin{bmatrix} \vdots \\ Y(\overline{q})^T \left[\left[\frac{\partial z_h^s}{\partial m_h^s} - \frac{\partial z_1^c}{\partial m_1^s}\right]^T \widetilde{\alpha}^s\right] \\ \vdots \end{bmatrix}. \end{aligned}$$

By assumptions, for each h, there are, at most, $(J-1)$ non-zero coordinates of the vector $\left[..., \sum_{c>1}\left(\frac{\partial z_h^{sc}}{\partial m_h^s} - \frac{\partial z_1^{sc}}{\partial m_1^s}\right)\widetilde{\alpha}^{sc}, ...\right]$. Since $Y(q)$ is in general position, this implies that $\left[G(.)\right]^T \widetilde{\alpha}^T = 0$ if and only if $\left[..., \sum_{c>1}\left(\frac{\partial z_h^{sc}}{\partial m_h^s} - \frac{\partial z_1^{sc}}{\partial m_1^s}\right)\widetilde{\alpha}^{sc}, ...\right] = 0$, for each s and h.

To conclude, we will now show that, for each s and h, $\left[..., \sum_{c>1}\left(\frac{\partial z_h^{sc}}{\partial m_h^s} - \frac{\partial z_1^{sc}}{\partial m_1^s}\right)\widetilde{\alpha}^{sc}, ...\right] = 0$ if and only if $\widetilde{\alpha}^s = 0$. Since, by assumption, $||\widetilde{\alpha}|| = 1$, this is impossible. To establish this last step, we make appeal, once again, to the transversality thm. applied to the following system of equations: for given s, and $\theta \in S^{C-2}$,

$$\Psi^s\left(\overline{p}, \overline{q}, \alpha; \omega, u\right) = \begin{bmatrix} \Phi(\overline{p}, \overline{q}; \omega, u) \\ \\ \left[..., \nabla_{m_h^s}^s z_h^s - \nabla_{m_1^s} z_1^s, ...\right]^T \theta^T \end{bmatrix} = 0.$$

Its derivative contains

$$D_{(\omega,\overrightarrow{du})}\Psi^s(\overline{p},\overline{q},\alpha;\omega,u)=\begin{bmatrix} D_\omega\Phi(.) & D_{\overrightarrow{du}}\Phi(.) \\ & \\ * & D_{\overrightarrow{du}}\left[\left[...,\nabla_{m_h^s}z_h^s-\nabla_{m_1^s}z_1^s,...\right]^T\theta^T\right]\end{bmatrix}=0.$$

We apply the same type of utility perturbations as above, so that $D_{\overrightarrow{du}}\Phi(.)=0$. Since $\theta\in S^{C-2}$, there is c such that $\theta^c\neq 0$. Perturbing, for each agent $h>1$, $\frac{\partial z_h^{sc}}{\partial m_h^s}$, we obtain

$$D_{\overrightarrow{du}}\left[\left[...,\left[\nabla_{m_h^s}z_h^s-\nabla_{m_1^s}z_1^s\right]^T,...\right]^T\theta^T\right]=\theta^c I.$$

Hence, $\Psi^s\pitchfork 0$ at each $(\overline{p},\overline{q},\alpha;\omega,u)\in\Psi^{s-1}(0)$. By transversality, for a generic subset $\mathcal{E}^s\subset\mathcal{E}^{gp}$, $\Psi^s_{(\omega,u)}\pitchfork 0$. Since $\Psi^s_{(\omega,u)}:\mathbb{R}^{(S+1)(C-1)+J}\times S^{C-2}\to\mathbb{R}^{(S+1)(C-1)+J}\times\mathbb{R}^{(H-1)}$, and $(H-1)>(C-2)$, $\Psi^s_{(\omega,u)}\pitchfork 0$ means $\Psi^{s-1}_{(\omega,u)}(0)=\emptyset$.

This concludes the proof: for each s, there is an open and dense set $\mathcal{E}^s\subset\mathcal{E}^{gp}$ such that $\left[...,\left[\nabla^s_{m_h^s}z_h^s-\nabla_{m_1^s}z_1^s\right]^T,...\right]$ has full row rank $(C-1)$. By taking intersection over s, we construct a generic set $\mathcal{E}^r\subset\mathcal{E}^{gp}$ such that, for each economy $(\omega,u)\in\mathcal{E}^r$, rank$\left[...,\left[\nabla^s_{m_h^s}z_h^s-\nabla_{m_1^s}z_1^s\right]^T,...\right]=(C-1)$ for each s. As argued above, this implies that case (b) is impossible.

Hence, $\Theta_{(\omega,u)}(\overline{p},\overline{q},\alpha)\pitchfork 0$, which implies that, at each equilibrium, the matrix $G(.)$ has maximal rank $(S+1)(C-1)$. This establishes iv. ■

Proof of Lemma 6. Regularity and strong regularity at each no-trade equilibrium hold because of the absence of income effects.

To establish the second part of the Lemma, let's first consider no-trade, Pareto inefficient equilibria. For future reference, we start showing a result which is stronger than the one claimed in this Lemma:

Fact. Given $\overline{u}$, consider any $\overline{\omega}\in\Omega^{NT}(K)$ and any open set $B(\overline{\omega},\overline{u})$. Then, there is some $(\omega,u)\in B(\overline{\omega},\overline{u})$ with a regular equilibrium such that, for some $\overrightarrow{dp}$, both $\overrightarrow{dp}\Lambda(\lambda,z)>>0$.

Proof. Modulo a locally linear perturbation of the utility functions and a relabelling of agents, we can assume that, at the no-trade equilibrium, the normalized Lagrange multipliers satisfy $\overline{\lambda}_1^{n1}>...>\overline{\lambda}_H^{n1}$. Consider the matrix $\Lambda(\lambda,z)\equiv$

$$\begin{bmatrix} \begin{bmatrix}-1\\ \vdots\\ 0\end{bmatrix}\delta & \begin{bmatrix}\frac{1}{H-1}\\ \vdots\\ 0\end{bmatrix}\delta & \cdots & \begin{bmatrix}\left[-\frac{1}{H-1}\right]\\ \vdots\\ 0\end{bmatrix}\delta \\ \begin{bmatrix}\overline{\lambda}_1^{n1}\left(\frac{1+\varepsilon\overline{\lambda}_1^{n1}}{\overline{\lambda}_1^{n1}}\right)\\ \vdots\\ 0\end{bmatrix}\delta & \begin{bmatrix}-\overline{\lambda}_2^{n1}\frac{\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon}{H-1}\\ \vdots\\ 0\end{bmatrix}\delta & \cdots & \begin{bmatrix}-\overline{\lambda}_H^{n1}\frac{\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon}{H-1}\\ \vdots\\ 0\end{bmatrix}\delta \\ [0] & [0] & \cdots & [0]\end{bmatrix},$$

with non-zero excess demand just for commodity 2 at $s=0,1$. Evidently, for $(\delta,\varepsilon)>>0$ and sufficiently small,

$$[1,...,1]\Lambda(\lambda,z)=\left[\overline{\lambda}_1^{n1}\varepsilon\delta,\frac{\frac{\overline{\lambda}_1^{n1}-\overline{\lambda}_2^{n1}}{\overline{\lambda}_1^{n1}}-\overline{\lambda}_2^{n1}\varepsilon}{H-1}\delta,...,\frac{\frac{\overline{\lambda}_1^{n1}-\overline{\lambda}_H^{n1}}{\overline{\lambda}_1^{n1}}-\overline{\lambda}_H^{n1}\varepsilon}{H-1}\delta\right]>>0.$$

Given $(\overline{\omega},\overline{u})$, define $(\omega,\overline{u})$ as follows:

1. for $h=1$, $\omega_1^{sc}=\overline{x}_1^{sc}$ for $sc\neq 01,02,11,12$. $\omega_1^{02}=\overline{x}_1^{02}+\delta$, $\omega_1^{01}=\overline{x}_1^{01}-p^{02}(\overline{\omega})\delta$, $\omega_1^{12}=\overline{x}_1^{12}-\delta\left[\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon\right]$, $\omega_1^{11}=\overline{x}_1^{11}+p^{02}(\overline{\omega})\delta\left[\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon\right]$,
2. for $h>1$, $\omega_h^{sc}=\overline{x}_h^{sc}$ for $sc\neq 01,02,11,12$. $\omega_h^{02}=\overline{x}_h^{02}-\frac{\delta}{H-1}$, $\omega_h^{01}=\overline{x}_h^{01}+p^{02}(\overline{\omega})\frac{\delta}{H-1}$, $\omega_h^{12}=\overline{x}_h^{12}+\delta\left[\frac{\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon}{H-1}\right]$, $\omega_h^{11}=\overline{x}_h^{11}-p^{02}(\overline{\omega})\delta\left[\frac{\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon}{H-1}\right]$.

We now show that $(\overline{x},p(\overline{\omega}),q(\overline{\omega}))$ is an equilibrium of $(\omega,\overline{u})$ with excess demand $z_h^{sc}=0$ for $sc\neq 01,02,11,12$, for each h, and

$$\left[z_1^{02},z_1^{12}\right]=\left[\delta,-\delta\left[\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon\right]\right],\qquad\left[z_h^{02},z_h\right]=\left[\frac{-\delta}{H-1},\frac{\delta}{H-1}\left[\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon\right]\right].$$

Evidently, market clears. We just need to show that, for each agent, given $(\omega_h,\overline{u}_h)$, $\overline{x}_h$ is the optimal choice at prices $(p(\overline{\omega}),q(\overline{\omega}))$. Since, by construction, $\nabla_{x_h^s}u_h(x_h)=\overline{\lambda}_h^s p^s(\overline{\omega})$, for each s, and $\overline{\lambda}_h^s Y(q(\overline{\omega}))=0$, it suffices to show that $\overline{x}_h$ is budget feasible for each h. By construction,

$$\begin{aligned}p^0(\overline{\omega})\omega_h^0 &= \overline{\omega}_1^{01}-\delta p^{02}(\overline{\omega})+p^{02}(\overline{\omega})\left[\overline{\omega}_1^{02}+\delta\right]+\sum_{c>2}p^{0c}(\overline{\omega})\overline{\omega}_1^{0c}=p^0(\overline{\omega})\overline{x}_1^0,\\ &\text{and}\\ p^1(\overline{\omega})\omega_h^1 &= \overline{\omega}_1^{11}+\delta\left[\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon\right]p^{12}(\overline{\omega})+p^{12}(\overline{\omega})\left[\overline{\omega}_1^{12}-\delta\left[\frac{1}{\overline{\lambda}_1^{n1}}+\varepsilon\right]\right]\\ +\sum_{c>2}p^{1c}(\overline{\omega})\overline{\omega}_1^{1c} &= p^1(\overline{\omega})\overline{x}_1^1.\end{aligned}$$

Similarly, for $h>1$. Hence, for each h, $\overline{x}_h$ is the optimal consumption bundle at prices $(p(\overline{\omega}),q(\overline{\omega}))$, given $(\omega_h,\overline{u}_h)$, so that $(p(\overline{\omega}),q(\overline{\omega}))$ with allocation $\overline{x}$ is an equilibrium of $(\omega_h,\overline{u}_h)$. By construction, $\overrightarrow{dp}=[1,0,...,0,1,0,...,0]$ satisfies $\overrightarrow{dp}\Lambda(.)>>0$. Hence, $\overrightarrow{dp}D_{\widetilde{p}}\widetilde{V}_h>0$, for each h. ■

To conclude the proof of the Lemma, observe that, by the same argument used to establish Thm. 2 iv., $D_{\widetilde{b}}\widehat{Z}$ has full row rank, so that $\left[D_{\widetilde{b}}\widehat{Z},D_{\widetilde{p}}\widehat{Z}\right]$ has full row rank. Moreover, for each $\overrightarrow{dp}$, there is $\overrightarrow{db}$ such that $\left[D_{\widetilde{b}}\widehat{Z}\right]\overrightarrow{db}+\left[D_{\widetilde{p}}\widehat{Z}\right]\overrightarrow{dp}=0$. This implies that MFCQ holds. ■

Proof of Proposition 12. We will construct an open set of economies with a unique CPO equilibrium, in some neighborhood of an economy $(\omega^{PO},\overline{u})$ such that preferences are separable across periods and states, identical across agents and homothetic. The endowment is PO. Thus, the unique equilibrium allocation of $(\omega^{PO},\overline{u})$ is no-trade and PO.

We split the proof into two steps.

Step 1. There is an open neighborhood $B_{2\varepsilon}(\omega^{PO},\overline{u})$, $\varepsilon>0$, such that, for each $(\omega^{PO},\overline{u})\in B_{2\varepsilon}(\omega^{PO},\overline{u})$, there is a unique equilibrium which is CPO with repect to each $\widetilde{b}\in B_{2\xi}\left(\overline{b}(\omega^{PO},\overline{u})\right)$, for some $\xi>0$.

Proof of Step 1. Evidently, there is some open neighborhood $B_{3\varepsilon}(\omega^{PO},\overline{u})$ such that the actual equilibrium $(\overline{p}(\omega,u),\overline{q}(\omega,u))$ is unique and regular. Also, each conditional equilibrium is strongly regular for each $(\omega, u, \widetilde{b}) \in B_{3\varepsilon}\left(\omega^{PO}, \overline{u}\right) \times B_{3\xi}\left(\overline{b}(\omega^{PO}, \overline{u}) = 0\right)$, for some $\xi > 0$. Moreover, there is no loss of generality in assuming that, for each h, diag$(\frac{\partial \widetilde{\lambda}_h^s(\widetilde{b},\omega,u)}{\partial m^s})$ is negative-definite on the same set. Indeed, if u_h is homogeneous of degree 1 in the income vector, replace it with $\overline{u}_h \equiv (u_h)^{\alpha}$, $\alpha \in (0,1)$. This has no effect on the demand functions, and their derivatives, while it affects the gradient of the vector of Lagrange multipliers. Given any open neighborhood $B_{3\varepsilon}(\omega^{PO}, u)$, we can choose α close enough to 1, so that $(\omega^{PO}, \overline{u}) \in B_{3\varepsilon}(\omega^{PO}, u)$. Then, since preferences are separable, the matrix $D_{m_h}\widetilde{\lambda}_h(\widetilde{b}, \omega, u)$ is diagonal and negative-definite.

Restrict the analysis to some open set of economies in $B_{3\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, say $B_{2\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, such that $||\overline{b}(\omega, u)|| \leq \xi$. Then, for each $(\omega, u) \in B_{2\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, if $\widetilde{b} \in B_{\xi}\left(\overline{b}(\omega, u)\right)$, $||\widetilde{b}|| \leq ||\overline{b}(\omega, u)|| + ||\overline{b}(\omega, u) - \widetilde{b}|| < 2\xi$. Thus, given any $(\omega, u) \in B_{2\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, if $\widetilde{b} \in B_{2\xi}\left(\overline{b}(\omega, u)\right)$, diag$(\frac{\partial \widetilde{\lambda}_h^s(\widetilde{b},\omega,u)}{\partial m^s})$ is negative-definite and the $\widetilde{b}-$conditional equilibrium is strongly regular.

Given $(\omega, u) \in B_{2\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, restrict the analiysis to $\widetilde{b} \in B_{\xi}\left(\overline{b}(\omega, u)\right)$.

The effect of a feasible portfolio adjustment $\overrightarrow{db} \equiv [db_1, ..., db_H] \equiv \left[..., \widetilde{b}_h - \overline{b}_h, ...\right]$ on agent h's utility can be computed using a second order Taylor's expansion

$$M_h(\widetilde{b}, \omega, u) \equiv \widetilde{V}_h(\widetilde{p}(\widetilde{b}, .), \overline{q}\,(.)\,, \widetilde{b}, \omega, u) - V_h(\overline{p}(.), \overline{q}\,(.)\,, \omega, u) = F_h(\widetilde{b}, \omega, u) + \frac{S_h(\widetilde{b}, \omega, u, \eta)}{2},$$

where $F_h(\widetilde{b}, \omega, u)$ denotes the first order effect, while $S_h(\widetilde{b}, \omega, u, \eta)$ measures the second order effect, evaluated at the equilibrium associated with $b^{\eta} = \overline{b} + \eta\overrightarrow{db}$. Bear in mind that, by definition, $M_h(\widetilde{b}, \omega, u)$ incorporates both direct and indirect effects of $\overrightarrow{db}$ on equilibrium utilities, i.e., it also reflects the impact of the induced price adjustments. By direct computation,

$$F_h(\widetilde{b}, \omega, u) = [db_h]\,[Y(\overline{q})]^T\,\left[\overline{\lambda}_h\right]^T - \left[\overline{\lambda}_h^0 z_h^0, ..., \overline{\lambda}_h^S z_h^S\right]\left[D_{\widetilde{b}}\widetilde{p}\,(.)\right]^T\left[\overrightarrow{db}\right]^T.$$

Evaluated at some b^{η}, $\eta \in [0,1]$,

$$\begin{aligned}
S_h(\widetilde{b}, \omega, u, \eta) \quad = \quad & [db_h]\,[Y(\overline{q})]^T\left[diag\left(\frac{\partial\lambda_h^s(\widetilde{b}, \omega, u, \eta)}{\partial m^s}\right)\right][Y(\overline{q}\,(\omega, u))]\,[db_h]^T \\
& + [db_h]\,[Y(\overline{q})]^T A_h(\widetilde{b}, \omega, u, \eta)\left[D_{\widetilde{b}}\widetilde{p}(\widetilde{b}, \omega, u, \eta)\right]^T\left[\overrightarrow{db}\right]^T \\
& + \left[\overrightarrow{db}\right]\left[D_{\widetilde{b}}\widetilde{p}(\widetilde{b}, \omega, u, \eta)\right]B_h(\widetilde{b}, \omega, u, \eta)\left[D_{\widetilde{b}}\widetilde{p}(\widetilde{b}, \omega, u, \eta)\right]^T\left[\overrightarrow{db}\right]^T \\
& + \left[\overline{\lambda}_h^0 z_h^0, ..., \overline{\lambda}_h^S z_h^S\right]\left[\begin{array}{c} \vdots \\ \left[\overrightarrow{db}\right]\left[D_{\widetilde{b}}^2\widetilde{p}^{sc}\,(.)\right]\left[\overrightarrow{db}\right]^T \\ \vdots \end{array}\right].
\end{aligned}$$

The first term is a negative definite matrix. $A_h(\widetilde{b}, \omega, u, \eta)$ and $B_h(\widetilde{b}, \omega, u, \eta)$ are well-defined matrices. Their coefficients are continuous functions, because, in particular, the matrix $D_{\widetilde{p}}\widetilde{Z}\,(.)$ has, locally, full rank.

When preferences are identical and homothetic, since $\widetilde{p}(b^{\eta}, \omega, u)$ is $b^{\eta}-$invariant, at each equilibrium, all the terms of $S_h(\widetilde{b}, \omega, u, \eta)$, but the first, are nil. Therefore, for each feasible $(\widetilde{b}, \omega, u, \eta)$ with $\widetilde{b}_h \neq \overline{b}_h$, $S_h(\widetilde{b}, \omega, u, \eta) < 0$. We will now show that a similar property holds for all the economies in a neighborhood of $(\omega^{PO}, \overline{u})$.

Let $\overrightarrow{d}^{\backslash} \equiv [d_2, ..., d_H] \in S_{\xi}^{(H-1)J-1}(\omega, u)$, the sphere in $\mathbb{R}^{(H-1)J}$ of radius ξ centered on $\overline{b}(\omega, u) =$

0. Let $\overrightarrow{d} \equiv [-\sum_{h>1} d_h, \overrightarrow{d^{\setminus}}]$.

Given (ω, u), define the problem

$$\max S_h(\widetilde{b}, \omega, u, \eta) \text{ subject to } (\overrightarrow{d^{\setminus}}, \eta) \in S_{\xi}^{(H-1)J-1}(\omega, u) \times [0,1].$$

Evidently, there is an optimal solution $(\overrightarrow{d^{\setminus *}}, \eta^*)(\omega, u)$ for each $(\omega, u) \in B_{2\varepsilon}(\omega^{PO}, \overline{u})$, and, by Berge's thm., the associated value of $S_h(.)$, $S_h^*(\omega, u)$, is a continuous function. Evidently, $S_h^*(\omega^{PO}, \overline{u}) \leq \chi < 0$, for each h. Thus, there exists an open neighborhood, say $B_\varepsilon(\omega^{PO}, \overline{u}) = \cap_h S_h^{*-1}(-\infty, \frac{\chi}{2}) \cap B_{2\varepsilon}(\omega^{PO}, \overline{u})$, such that, for each $(\omega, u) \in B_\varepsilon(\omega^{PO}, \overline{u})$, $S_h^*(\omega, u) \leq \frac{\chi}{2} < 0$, for each h. Hence, for each $(\omega, u) \in B_\varepsilon(\omega^{PO}, \overline{u})$, $S_h(.) \leq \frac{\chi}{2} < 0$ for each $\overrightarrow{d^{\setminus}} \in S_{\xi}^{(H-1)J-1}$, $\eta \in [0,1]$, and each h.

For now, we only compare the actual equilibrium to conditional equilibria associated with $\widetilde{b}$ such that $||(\widetilde{b}_2, ..., \widetilde{b}_H) - (\overline{b}_2(\widetilde{\omega}, \widetilde{u}), ..., \overline{b}_H(\widetilde{\omega}, \widetilde{u}))|| \leq \xi$. Pick any feasible $\widetilde{b}$ and consider $M_h(\widetilde{b}, \widetilde{\omega}, \widetilde{u})$. By construction, $\widetilde{b} = \overline{b} + \widetilde{\theta}\overrightarrow{db}$, for some $\widetilde{\theta} \in [0,1]$. Thus, for each h, $S_h(\widetilde{b}, \widetilde{\omega}, \widetilde{u}, \eta) < 0$, while

$$F_h(\widetilde{b}, \widetilde{\omega}, \widetilde{u}) = \widetilde{\theta} \underbrace{[db_h] [Y(\overline{q})]^T \left[\overline{\lambda}_h\right]^T}_{=0} - \widetilde{\theta} \left[\overline{\lambda}_h^0 \overline{z}_h^0, ..., \overline{\lambda}_h^S \overline{z}_h^S\right] \left[D_{\widetilde{b}} \widetilde{p}(.)\right] \left[\overrightarrow{db}\right]^T.$$

Hence, if $\widetilde{b}$ Pareto improves upon $\overline{b}$, the induced price adjustment must satisfy $\left[D_{\widetilde{b}} \widetilde{p}(.)\right] \left[\overrightarrow{dp}\right]^T << 0$. By Stiemke thm. of the alternatives,

$$(A)\ \phi \begin{bmatrix} \vdots \\ \overline{\lambda}_h^0 \overline{z}_h^0, ..., \overline{\lambda}_h^S \overline{z}_h^S \\ \vdots \end{bmatrix} = 0 \qquad \text{or} \quad (B) \quad \begin{bmatrix} \vdots \\ \overline{\lambda}_h^0 \overline{z}_h^0, ..., \overline{\lambda}_h^S \overline{z}_h^S \\ \vdots \end{bmatrix} \left[\overrightarrow{dp}\right]^T << 0,$$

for some $\phi \geq 0$, but never both. To conclude, it suffices to show that such a positive ϕ exists.

Pick any economy $(\widetilde{\omega}, \widetilde{u}) \in B_\varepsilon(\omega^{PO}, \overline{u})$ with the following properties:

1. $\left[..., \left[D_{\widetilde{b}_h} \widetilde{z}_h(\overline{p}(.), \overline{q}(.)) - D_{\widetilde{b}_1} \widetilde{z}_1(\overline{p}(.), \overline{q}(.), \widetilde{b}_1)\right]^T, ...\right]$ has full row rank (hence preferences are not anymore h-invariant),
2. $\overline{x}(\overline{p}(\widetilde{\omega}, \widetilde{u}), \overline{q}(\widetilde{\omega}, \widetilde{u}), \widetilde{\omega}, \widetilde{u})$ is not PO,
3. there is a strictly positive solution ϕ to (A), for $h = 1, ...H$.

First, observe that there are open sets of economies close to $\left(\omega^{PO}, \overline{u}\right)$ such that (3) holds. Indeed, given $\left(\omega^{PO}, \overline{u}\right)$, perturb the initial endowment to some ω', so that the matrix with typical row $\left[\overline{\lambda}_h^0 \left(\omega_h^{0\prime} - \omega_h^{0PO}\right), ..., \overline{\lambda}_h^S \left(\omega_h^{S\prime} - \omega_h^{SPO}\right)\right]$ has now full column rank, while ω^{PO} is still the equilibrium allocation, with $\overline{b}_h(\overline{p}(\omega^{PO}, \overline{u}), \overline{q}(\omega^{PO}, \overline{u}), \omega', \overline{u}) = 0$. Set $\overline{\phi}_h = \frac{\overline{\lambda}_1^0}{\overline{\lambda}_h^0}$, for each h. Since Pareto optimality implies collinearity of $\overline{\lambda}_h$ and $\overline{\lambda}_{h'}$ for each h, h', by market clearing, condition (A) above is satisfied for $(\omega', \overline{u})$. Then, it suffices to pick $(\omega", u") \in B_\varepsilon(\omega^{PO}, \overline{u})$, satisfying (1) and (2) above and close to $(\omega', \overline{u})$, to guarantee that there is some $\phi" >> 0$ such that (A) holds at $(\omega", u")$. It follows that, for such an economy, there is no $\overrightarrow{dp}$ such that

$$\left[..., \overline{\lambda}_h^s(\omega", u") \overline{z}_h^s(\omega", u"), ...\right] \left[\overrightarrow{dp}\right]^T > 0,$$

each h. Hence, for each $\widetilde{b}$, there is at least one agent h such that both $S_h(\widetilde{b}, \omega, u, \eta) < 0$ and $F_h(\widetilde{b}, \omega, u) < 0$. Thus, for some agent,

$$\widetilde{V}_h(\widetilde{p}(\widetilde{b}, .), \overline{q}\,(.)\,, \widetilde{b}, \omega, u) - V_h(\overline{p}(.), \overline{q}\,(.)\,, \omega, u) < 0.$$

Therefore, there is no $\widetilde{b} \in S_{\xi}^{(H-1)J-1}\,(\omega", u")$ Pareto improving upon the equilibrium allocation.
■

This does not conclude the argument. In principle, we could still have that equilibria are dominated by conditional equilibria associated with some $\widetilde{b}$ such that the induced $\widetilde{\theta}\overrightarrow{db} \notin intS_{\xi}^{(H-1)J-1}\,(\omega, u)$.

Step 2. There is an open neighborhood $B_{\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, $\varepsilon > 0$, such that, for each $\left(\omega^{PO}, \overline{u}\right) \in B_{\varepsilon}\left(\omega^{PO}, \overline{u}\right)$, the unique equilibrium is CPO.

Proof of Step 2. Bear in mind that all the properties specified above in the construction of $(\omega", u")$ are open. Suppose that there is no open neighborhood of $\left(\omega^{PO}, \overline{u}\right)$ such that all the equilibria of economies satisfying $(1-3)$ are CPO. Then, we can construct a sequence $\{(\omega^v, u^v)\}_{v=1}^{\infty}$, $(\omega^v, u^v) \to \left(\omega^{PO}, \overline{u}\right)$ of economies satisfying $(1-3)$ and such that, for each v, there is a $\widetilde{b}^v$ such that the associated equilibrium Pareto dominates $\overline{x}\left(\overline{p}\left(\omega^v, u^v\right), \overline{q}\left(\omega^v, u^v\right), \omega^v, u^v\right)$, i.e., for each v, and each h, $M_h(\widetilde{b}^v, \omega^v, u^v) \geq 0$. Since all sequences can be taken to be convergent, $\widetilde{b}^v \to b^{\circ}$, $\widetilde{p}(\widetilde{b}^v, \omega^v, u^v) \to \widetilde{p}^{\circ}$, and $\widetilde{x}^v \to x^{\circ}$, we conclude that

$$\widetilde{V}_h(\widetilde{p}^{\circ}, \overline{q}\left(\omega^v, u^v\right), \widetilde{b}^{\circ}, \omega^{PO}, \overline{u}) - V_h(\overline{p}\left(\omega^{PO}, \overline{u}\right), \overline{q}\left(\omega^{PO}, \overline{u}\right), \left(\omega^{PO}, \overline{u}\right)) \geq 0.$$

Since, for each v, $||\widetilde{b}^v - \overline{b}^v|| \geq \xi$, it must be $||\widetilde{b}^{\circ}|| \geq \xi$. It is easy to check that, given that R has full rank, this implies $x^{\circ} \neq \omega^{PO}$. This is impossible because utility functions are strictly quasi-concave: $x^{\circ} \neq \omega^{PO}$, for each $\pi \in [0,1]$ $x^{\pi} = \pi\omega^{PO} + (1-\pi)x^{\circ}$ is feasible and strictly Pareto superior to ω^{PO}. This cannot be, because ω^{PO} is PO. ■

References

[1] Citanna, A., A. Kajii, and A., Villanacci, 1998, Constrained Suboptimality in Incomplete Markets: A General Approach and Two Applications. Economic Theory 11, 495-521.

[2] Geanakoplos, J., and H.M. Polemarchakis, 1980, On the Disaggregation of Excess Demand Functions. Econometrica 48, 315-332.

[3] Geanakoplos, J., and H.M. Polemarchakis, 1986, Existence, Regularity and Constrained Suboptimality of Competitive Allocations When the Asset Market Is Incomplete. In W.P. Heller, R.M. Starr and D. Starrett (eds.), Uncertainty, Information and Communication: Essays in Honor of K.J. Arrow, Vol. III. Cambridge University Press, Cambridge, 65-95.

[4] Hart, O.J., 1975, On the Optimality of Equilibrium When the Market Structure Is Incomplete. J. Econ. Theory 11, 418-443.

[5] Janin, R., 1984, Directional Derivative of the Marginal Function in Nonlinear Programming. Math. Program. Study 21, 110-126.

[6] Kajii, A., 1992, Constrained Suboptimality with Many Agents. CARESS Working Paper #92-29, University of Pennsylvania.

[7] Mangasarian, O.L., and S. Fromovitz, 1967, The Fritz-John Necessary Optimality Conditions in Presence of Equality and Inequality Constraints, J. Math. Anal. Appl. 17, 37-47.

[8] Mas-Colell, A., 1987, On the Theorem of Geanakoplos-Polemarchakis. Unpublished manuscript, Harvard University.

[9] Mendolicchio, C., and T. Pietra, 2016, Endowment Redistribution and Pareto Improvements in GEI Economies. IAB Discussion Paper 1/2016.

[10] Nagata, R., 2005, Inefficiency of Equilibria with Incomplete Markets. Journal of Mathematical Economics 41, 887-897.

[11] Pietra, T., 2004, Sunspots, Indeterminacy and Pareto Inefficiency in Economies With Incomplete Asset Markets. Economic Theory 24, 687-699.

[12] Salto, M., and T. Pietra, 2013, Welfare and Excess Volatility of Exchange Rates. Economic Theory 52, 501–529.

[13] Stiglitz, J.E., 1982, The Inefficiency of the Stock Market Equilibrium. Review of Economic Studies 49, 241-261.

[14] Tinbergen, J., 1956, Economic Policy: Problems and Design, North-Holland, Amsterdam.

[15] Tirelli, M., 2008, Constrained Inefficiency in GEI: A Geometric Argument. Journal of Mathematical Economics 44, 1197–1214.

[16] Werner, J., 1991, On Constrained Optimal Allocations with Incomplete Markets. Economic Theory 1, 205-209.

Recently published

No.	Author(s)	Title	Date
32/2015	Fuchs, J. Weber, E.	Long-term unemployment and labor force participation: A decomposition of unemployment to test for the discouragement and added worker hypotheses	11/15
33/2015	Reichelt, M., Haas, A.	Commuting farther and earning more? How employment density moderates workers' commuting distance	11/15
34/2015	van den Berg, G. Uhlendorff, A. Wolff, J.	Under heavy pressure: Intense monitoring and accumulation of sanctions for young welfare recipients in Germany	12/15
35/2015	Kampkötter, P. Mohrenweiser, J. Sliwka, D. Steffes, S. Wolter, S.	Measuring the use of human resources practices and employee attitudes: The Linked Personnel Panel	12/15
1/2016	Mendolicchio, C. Pietra, T.	Endowment redistribution and Pareto improvements in GEI economies	1/16
2/2016	Van den Berg, G. Hofmann, B. Uhlendorff, A.	The role of sickness in the evaluation of job search assistance and sanctions	1/16
3/2016	Bossler, M.	Employment expectations and uncertainties ahead of the new German minimum wage	2/16
4/2016	Fuchs, J. Kubis, A. Schneider, L.	Replacement migration from a labour market perspective: Germany's long-term potential labour force and immigration from non-EU member countries	2/16
5/2016	Garloff, A. Wapler, R..	Labour shortages and replacement demand in Germany: The (non)-consequences of demographic change	2/16
6/2016	Garloff, A. Roth, D.	Regional age structure and young workers' wages	2/16
7/2016	Stockinger, B. Wolf, K.	The productivity effects of worker mobility between heterogeneous firms	2/16
8/2016	Bruckmeier, K. Wiemers, J.	Differences in welfare take-up between immigrants and natives: A microsimulation study	3/16
9/2016	Möller, J.	Lohnungleichheit - Gibt es eine Trendwende?	3/16
10/2016	Bossler, M. Gerner, H.-D.	Employment effects of the new German minimum wage: Evidence from establishment-level micro data	3/16
11/2016	Bossler, M. Grunau, P.	Asymmetric information in external versus internal promotions	4/16

As per: 2016-04-14

For a full list, consult the IAB website
http://www.iab.de/de/publikationen/discussionpaper.aspx

Imprint

IAB-Discussion Paper 12/2016
20 April 2016

Editorial address
Institute for Employment Research
of the Federal Employment Agency
Regensburger Str. 104
D-90478 Nuremberg

Editorial staff
Ricardo Martinez Moya, Jutta Palm-Nowak

Technical completion
Gertrud Steele

Website
http://www.iab.de

Download of this Discussion Paper
http://doku.iab.de/discussionpapers/2016/dp1216.pdf

ISSN 2195-2663

For further inquiries contact the author:

Concetta Mendolicchio
Phone +49.911.179 2330
E-mail concetta.mendolicchio@iab.de