Institute for Employment Research
The Research Institute of the Federal Employment Agency

IAB

IAB-Discussion Paper 1/2016

Articles on labour market issues

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ISSN 2195-2663

Endowment redistribution and Pareto improvements in GEI economies

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Abstract

With incomplete markets and numeraire assets, there are open sets of economies such that their equilibrium allocations can be improved upon by a reallocation of period zero endowments. This strengthens the classical results on constrained Pareto inefficiency of equilibria in GEI.

Zusammenfassung

Bei unvollkommenen Märkten und numerairen Vermögen gibt es eine offene Menge an Volkswirtschaften, so dass die Gleichgewichtsverteilung durch eine veränderte Anfangsverteilung verbessert werden kann. Dies stärkt das klassische Ergebnis der eingeschränkten Pareto Ineffizienz von Gleichgewichten in GEI.

JEL classification: D51, D52

Keywords: GEI, constrained Pareto optimality.

1. Introduction

In the absence of completeness of financial markets, equilibrium allocations are typically Pareto inefficient. In fact, the set of equilibrium allocation itself may be Pareto ranked, completely, as in the Hart (1975) example, or partially, as in Pietra (2004) and Salto and Pietra (2013).[1] In economies with real assets, however, Pareto ranking of equilibria is the exception, and it becomes important to formulate an appropriate efficiency criterion. The canonical definition of constrained Pareto optimality (CPO) has been introduced by Stiglitz (1982) and developed by Geanakoplos and Polemarchakis (1986) and Citanna, Kajii and Villanacci (1998). It rests on the idea that the minimal efficiency requirement an equilibrium allocation should satisfy is that it cannot be improved upon by a reallocation of asset holdings, and by the adjustment of prices required to restore the equilibrium in the commodity markets. Adopting the convenient fiction of a benevolent planner, this notion of CPO endows her with fairly limited instruments and, most important, it allows her to affect directly the intertemporal allocation of individual incomes using only the opportunities offered by the set of available assets. The possibility of improving upon the equilibrium allocation using portfolio reallocations rests on the welfare effects of the induced changes in equilibrium prices.

Different notions of constrained efficiency can be developed, in much the same spirit, by choosing other policy instruments. Herings and Polemarchakis (2004) show that, under suitable regularity conditions, price regulation can attain a Pareto improvement over fix-price equilibria. Citanna, Polemarchakis and Tirelli (2006) show that taxation of asset trades may also allow for Pareto improvements.

Here, we consider an alternative notion, which allows the planner to reallocate incomes just in the initial trading period, letting the agents choose their individually optimal portfolios and consumption bundles at the new equilibrium prices. The basic idea behind the canonical Geanakoplos and Polemarchakis's criterion and the one proposed here is the same: the planner chooses the value of a policy instrument, allows people to choose their optimal behavior, and adjusts prices to restore market clearing. Evidently, to use period 0 endowment reallocations as policy tools is fully coherent with the absence of some assets.[2] In GEI models, the market failure is due to the distorted intertemporal allocation of incomes induced by the absence of some assets. Hence, it could appear to be harder to improve upon the equilibrium allocations by just reallocating endowments at time 0. However, we show that there are open sets of economies such that this can be obtained, so that, for these economies, equilibria are not constrained efficient according to our criterion. Clearly, a key role is played by the choice of the specific vector of lump-sum taxes, so that the policy intervention is not anonymous (see Kajii (1994)).

There are several motivations for this paper. Its core issue - "can we improve upon a GEI equilibrium allocation by reallocating just period 0 endowments?" - has been around for a long time. It looks interesting to settle it. Our answer is only partially positive. Indeed, we show that, first, given any specification of an economy in terms of numeraire asset structure and preferences, there are open sets of endowments such that equilibria cannot be improved upon by reallocating period zero endowments. Secondly, there are open sets of economies where an appropriate period 0 endowment reallocation induces a Pareto improvement. The set is open in the space of economies defined in terms of asset structure, preferences and endowments. The result holds even if we restrict the analysis to time separable VNM utility functions.

1 These last two papers deal with economies with nominal asset and indeterminate equilibria. Under appropriate restrictions, generically each equilibrium allocation is Pareto inferior to some other equilibrium allocation.

2 Evidently, it is essential that the income transfers take place just in period zero. If we were to allow them in different spots, we would implicitly allow the planner to manufacture personalized assets, so that she could actually attain full Pareto optimality.

We believe that this second result is of interest for at least two additional reasons. First, this is, in a limited, but important, way, a counterexample to the claim that, to implement H policy aims, you need at least H independent policy instruments. As pointed out by Citanna et al. (1998), this viewpoint goes back to Tinbergen (1956). In our set up, there are H policy aims (the changes in the equilibrium utility of H agents) and $(H-1)$ independent instruments. Still, by properly exploiting the welfare effects of the induced price changes, we can attain a Pareto improvement for some open set of economies. Of course, the real issue is how we define a "policy aim." It is certainly true that, in general, at least H independent policy instruments are required to attain each specific vector of utility improvements, $du = (du_1, ..., du_H)$, so that this cannot be obtained by reallocating endowments at time zero. However, if you just aim to some $du > 0$, less than H policy tools may be enough. The point is that the result established in Geanakoplos and Polemarchakis (1986) and in Citanna et al (1998) is much stronger than what is strictly required to establish lack of CPO according to their notion. Their criterion simply requires that, by some policy intervention, we can attain some positive vector of changes in the equilibrium level of the utility of each agent. They, however, show that one can attain every positive vector. Not surprisingly, a stronger result requires stronger restrictions on the class of economies and of policy profiles than the ones minimally required. Secondly, the kind of policy intervention considered in Geanakoplos and Polemarchakis (1986) could suggest that the inefficiency associated with market incompleteness dictates Pareto improving measures related to interventions in the working of the financial markets - in their framework, to impose a portfolio to each agent - or to intertemporal policies. Our result shows that efficiency can be improved using just time 0 lump-sum taxes, without any intervention in specific markets.

The next section briefly presents the model. Section 3 formalizes our notion of constrained Pareto optimality and establishes our main results. Some conclusive remarks follow.

2. The Model

We consider a standard GEI model with numeraire assets. There is a finite set of agents $(h = 1, ..., H)$ and a finite set of commodities $(c = 1, ..., C)$ at each of $(S+1)$ spots, $s = 0, ..., S$. A consumption plan is $x_h \equiv \left(x_h^0, x_h^1, ..., x_h^S\right) \in \mathbb{R}_+^{(S+1)C}$, a portfolio is $b_h \equiv \left(b_h^1, ..., b_h^J\right) \in \mathbb{R}_+^J$. Commodity prices are $p \equiv \left(p^0, p^1, ..., p^S\right) \in \mathbb{R}_{++}^{(S+1)C}$, asset prices are $q \equiv \left(q^1, ..., q^J\right) \in \mathbb{R}^J$. As usual, we normalize to 1 the price of good 1 in each spot. Asset trade takes place at spot 0.[3] Asset payoffs are defined in terms of the numeraire commodity and described by a $(S \times J)$ matrix R of full rank

$$R = \begin{bmatrix} r^{11} & & r^{1J} \\ \vdots & \ddots & \vdots \\ r^{S1} & & r^{SJ} \end{bmatrix}.$$

Finally, $u_h(x_h)$ is agent h's utility function, satisfying the standard assumptions for the differential analysis of equilibria: for each h, $u_h(x_h)$ is C^2, strictly monotone, differentiably strictly quasi-concave in x_h, and satisfies the boundary conditions: the closure of the set $\{x_h : u_h(x_h) \geq u_h(\overline{x}_h)\}$ is contained in $\mathbb{R}_{++}^{S(C+1)}$, for each $\overline{x}_h \gg 0$.

Consumers' behavior is described by the optimal solution to the problem: Given (p, q), choose

$$(x_h, b_h) \quad \in \quad \arg\max u_h(x_h) \text{ subject to}$$

[3] A standard interpretation is that there are two periods and uncertainty on tomorrow state of the world. In view of the structure of some of the examples below, it is better to think of it as a multiperiod model, with or without uncertainty. The essential feature is that asset trade takes place just at time 0.

$$p^0\left(x_h^0 - \omega_h^0\right) \equiv p^0 z_h^0 = -q b_h, \qquad (U)$$
$$p^s\left(x_h^s - \omega_h^s\right) \equiv p^s z_h^s = r^s b_h, \text{ for each } s > 0,$$

where $\omega_h \equiv \left(\omega_h^0, \omega_h^1, ..., \omega_h^S\right) \in \mathbb{R}_{++}^{(S+1)C}$ is the initial endowment vector. Let $\lambda_h \in \mathbb{R}_{++}^{S+1}$ be the vector of Lagrange multipliers associated with the optimal solution to optimization problem (U). We do not impose that preferences can be described by a Von Neumann-Morgenstern utility function. However, our main results hold also for this more restricted class of economies.

Definition 1. A financial equilibrium is a price vector $(\overline{p}, \overline{q})$, with associated allocation and portfolio profiles $\left\{\left(\overline{x}_1, \overline{b}_1\right), ..., \left(\overline{x}_H, \overline{b}_H\right)\right\}$, such that:
a. for each h, $\left(\overline{z}_h, \overline{b}_h\right)$ solves problem (U) given $(\overline{p}, \overline{q})$,
b. $\sum_h \overline{z}_h = 0$ and $\sum_h \overline{b}_h = 0$.

3. Constrained suboptimality of equilibria

Let's briefly discuss the standard approach to the analysis of constrained suboptimality in GEI economies. Consider the system of eqs.

$$\Xi(p, q, \xi) = [\Phi(.), (..., u_h(.) - \overline{u}_h, ...)],$$

where $\Phi(.) = 0$ defines the equilibrium. The key step in the proof of constrained suboptimality is to show that $D_{(p,q,\xi)}\Xi(.)$ has, generically, full rank at each solution. This can be done using as equilibrium map $\Phi(.)$ the system of aggregate excess demand functions, as in Geanakoplos and Polemarchakis (1986), or the entire system of conditions (individual and aggregate) that an equilibrium must satisfy, as in Citanna et al. (1998).[4] Anyhow, the basic idea is the same: add to the equilibrium conditions the system of equations $[..., u_h(.) - \overline{u}_h, ...]$ and show that the map so obtained has a full rank derivative. This means that, by choosing appropriately the policy vector ξ, it is possible to implement every possible variation of the equilibrium level of the utility of each agent. If equilibria are locally determined, this approach necessarily requires that the number of degrees of freedom in the selection of the policy vector is at least as large as the number of agents, H. Hence, it cannot be applied to study the possibility of Pareto improvements obtained by a reallocation of period 0 endowments, since this policy instrument has, in an essential way, dimension $(H - 1)$.

However, what really matters is if it is possible to improve upon the equilibrium allocations, not the attainability of every possible Pareto improvement. From this viewpoint, the key issue is if the matrix $D_{(p,q,\xi)}\Xi(.)$ spans some non-trivial vector $[0, (..., du_h, ...)] \geq 0$, not if it spans all vectors with this structure. Evidently, by adopting this weaker condition, we could be able to weaken the restriction on the minimal rank of $D_{(p,q,\xi)}\Xi(.)$. In fact, as we will see, one of the robust examples provided below can also be seen as an example of an economy with just one asset, so that the dimension of the policy profile "portfolio reallocation" is smaller than H. The unique equilibrium is not CPO, according to the Geanakoplos and Polemarchakis (1986) criterion.

Here, however, we will focus on the possibility of Pareto improvements obtainable through the reallocation of the initial endowments of good 1 in period zero. i.e., our policy vector is a profile

[4] There are other differences between the two papers. In particular, in the second, the authors consider the welfare effect of a policy profile defined in terms of both portfolio reallocation and period 0 endowment reallocation. This is irrelevant for the purposes of the current discussion.

$t \equiv [t_1, ..., t_H]$ with $\sum_h t_h = 0$. Clearly, its dimension is $(H-1)$.[5] We now make precise our efficiency criterion.

Definition 2. A financial equilibrium $(\overline{p}, \overline{q})$ is $\omega-$Constrained Pareto Optimal ($\omega-$CPO) if there is no profile t with $\sum_h t_h = 0$ such that, at one associated equilibrium $(p(t), q(t))$, $u_h(x_h(p(t), q(t), t)) \geq u_h(x_h(\overline{p}, \overline{q}))$ for each h, with at least one strict inequality.

We start establishing the negative part of our result: given any profile of utility functions and any payoff matrix, there is an open set of economies such that all equilibria are $\omega-$CPO (but not necessarily CPO according to the Geanakoplos and Polemarchakis (1986) criterion). The argument is straightforward. Still, it may be worthwhile to elaborate a little on its logic before getting into the details.

To begin, for completeness, we report a standard result, i.e., the generalization of Roy' s identity to financial economies.

Lemma 3. Let $V_h(p, q)$ be the indirect utility function associated with optimization problem (U). Then,

$$\begin{aligned} \frac{\partial V_h}{\partial p^{sc}} &= -\lambda_h^s z_h^{sc}(p, q), \text{ for each } sc, \\ \frac{\partial V_h}{\partial q^j} &= -\lambda_h^0 b_h^0(p, q), \text{ for each } j. \end{aligned}$$

Proof. In Appendix. ∎

The effect of a change in portfolios and of the induced price changes on the utility of agent h can be decomposed into two parts: the direct effect of db_h on the indirect utility function, and the second order effect, due to the induced price changes. Let $\left[\overrightarrow{\frac{\partial G}{\partial t}x}\right]$ be the directional derivative of any function $G(.)$ in the direction $[x]$. By the noarbitrage conditions, $\frac{\partial V_h(p,q,b_h)}{\partial b_h^j} = 0$. Hence, just the second order effects of the portfolios reallocation matter. Using Roy's identity, they can be written as $\frac{\partial V_h(.)}{\partial p^{sc}} \left[\overrightarrow{\frac{\partial p^{sc}}{\partial b} db}\right] = -\lambda_h^s(.) z_h^{sc}(.) \left[\overrightarrow{\frac{\partial p^{sc}}{\partial b} db}\right]$, and $\frac{\partial V_h(.)}{\partial q^j} \left[\overrightarrow{\frac{\partial q^j}{\partial b} db}\right] = -\lambda_h^0(.) b_h^j(.) \left[\overrightarrow{\frac{\partial q^j}{\partial b} db}\right]$. To establish lack of CPO, it suffices that the span of the collection of these directional derivatives with respect to prices contains at least one strictly positive vector.

With our notion of $\omega-$CPO, the first order effect is not trivial. Indeed, it is $\frac{\partial V_h(.)}{\partial t_h} t_h = \lambda_h^0(.) t_h \neq 0$ and, evidently, there must be at least one agent h with $\frac{\partial V_h(p,q,b_h)}{\partial t_h} t_h < 0$. To obtain a Pareto improvement, the second order effects must have the right sign and, additionally, they must be sufficiently large, so that they can compensate the, possibly negative, first order effects for each agent. Given the formulas for $\left(\frac{\partial V_h(.)}{\partial p^{sc}}, \frac{\partial V_h(.)}{\partial q^j}\right)$ reported above, this can happen only if, at the initial equilibrium, the normalized vector of Lagrange multipliers are sufficiently different across agents, if net trades are sufficiently far away from 0, and/or if the directional derivatives of equilibrium prices are sufficiently large. This immediately rules out the possibility to Pareto improve upon the equilibria of economies with initial endowments close to a PO allocation. Therefore, there are open sets of economies with $\omega-$CPO equilibria. This argument is formalized in Proposition 4.

[5] We consider rellocations of good 1 endowments. Evidently, nothing would change by allowing for reallocation of the endowments of the other period 0 commodities.

Proposition 4. For each economy (u, R), there is an open set of endowments such that each equilibrium allocation is $\omega-$CPO.

Proof. Given (u, R), pick a Pareto optimal endowment profile, $\overline{\omega}$.[6] By a standard argument, the equilibrium is unique and regular. Moreover, for each agent, excess demand and portfolio are identically zero. Consider any endowment reallocation profile $\{\widehat{t}_1, ..., \widehat{t}_H\}$, with $\sum_h \widehat{t}_h = 0$. For each agent h,

$$\frac{1}{\overline{\lambda}_h^0}\left[\overrightarrow{\frac{\partial V_h}{\partial t}}\widehat{t}\right] = \widehat{t}_h - \sum_j \overline{b}_h \left[\overrightarrow{\frac{\partial q^j}{\partial t}}\widehat{t}\right] - \sum_{sc} \frac{\overline{\lambda}_h^s}{\overline{\lambda}_h^0} \overline{z}_h^{sc} \left[\overrightarrow{\frac{\partial p^{sc}}{\partial t}}\widehat{t}\right] = \widehat{t}_h.$$

Hence, for each $\widehat{t} \neq 0$, there is some h such that $\frac{1}{\overline{\lambda}_h^0}\left[\overrightarrow{dV_h \widehat{t}}\right] < 0$.

Since the original equilibrium is regular, the vector $\left[..., \left[\overrightarrow{\frac{\partial q^j}{\partial t}}\widehat{t}\right], ..., \left[\overrightarrow{\frac{\partial p^{sc}}{\partial t}}\widehat{t}\right], ...\right]$ is uniformly bounded above for each $t \in S^{H-1}$, the unit sphere. Hence, for ω close to $\overline{\omega}$, the equilibrium vector $\{..., (\overline{z}_h, \overline{b}_h), ...\}$ is close to zero. Therefore, by continuity, for each $\widehat{t}$, $\frac{1}{\overline{\lambda}_h^0}\left[\overrightarrow{\frac{\partial V_h}{\partial t}}\widehat{t}\right]$ is arbitrarily close to $\widehat{t}_h$ for ω close enough to $\overline{\omega}$. Hence, there is some open neighborhood of the endowment vector $\overline{\omega}$, $B(\overline{\omega})$, such that, for each $\omega \in B(\overline{\omega})$, there is a unique equilibrium and, at such an equilibrium, there is no $\widehat{t} \neq 0$ such that $\left[\overrightarrow{\frac{\partial V_h}{\partial t}}\widehat{t}\right] \geq 0$ for each h, i.e., for all the economies in this set, each equilibrium is $\omega-$CPO. ∎

Given our aim, this is a negative result. The main motivation for this paper is Prop. 7, showing that there are also open sets of economies with non $\omega-$CPO equilibria. We establish it by providing two parametric examples, and then showing that the same result holds for some open set of economies. We are not claiming that a similar result holds for some endowment profile given any specification of (u, R). It is fairly obvious that this cannot be true, for instance, for economies with identical, homothetic preferences.[7] Our result holds for some open set of economies in the space defined by endowments, utilities and payoffs. It is still an open issue if, given any (u, R) in some generic set, it holds for some appropriately chosen open set of endowments. We will come back to this issue in the conclusions.

The basic intuition for the possibility of a Pareto improvement can be most easily seen in an economy with just one asset, as in both examples: an endowment redistribution affects equilibrium prices for the asset and the commodities. Necessarily, within each spot the effect on the agents utilities must cancel out, since they are given by the product of the derivative of each price with respect to the endowment change multiplied by (minus) the excess demand for the commodity, or the asset holding. However, due to market incompleteness, for each agent, these changes in spot utilities are aggregated over spots using a distinct vector of normalized Lagrange multipliers so that the total utility changes do not necessarily cancel out when aggregated over the set of agents. Moreover, we need to take into account the direct effect of the time 0 transfer. Clearly, to provide an example of an economy with $\omega-$CP inefficient equilibria, we need to balance carefully the three different effects. At the same time, we need to maintain a structure simple enough, so that the computational burden is not too heavy. In both examples, we fix the class of economies and the endowment vectors. We then fix appropriately a price

[6] A similar argument works for each equilibrium with zero trade. We focus on equilibria with Pareto optimal initial endowments because this guarantees also uniqueness of the equilibrium.

[7] Since equilibrium prices are invariant with respect to the reallocation of spot income, our argument is bound to fail, as, in fact, does the one of Geanakoplos and Polemarchakis (1986).

vector and an allocation and choose the values of some parameters of the utility functions in such a way that the assigned price and allocation hold as an equilibrium. As we will see, this leaves us with enough degrees of freedom so that we can pick these parameters in such a way that the equilibrium is not $\omega - CPO$. We consider three period economies with one numeraire asset. Both examples are highly non generic. Still, all the equilibrium variables are, locally, continuous functions of the parameters, so that, as we will see, the main result is robust to open perturbations. The first, in the text, considers an economy with three agents, three spots and two consumption goods at each one of the future spots. The second (in Appendix) presents an economy with two agents, three periods and two goods just at one future spot. The two economies share many features. However, the first example requires computations which are somewhat more transparent, while the second shows that lack of CPO can hold even in two agent economies.

Example 5. There are three periods and just one asset, inside money, paying one unit of the numeraire commodity at each future period. To avoid unnecessary notation, we assume that there is just one commodity at time 0. This entails no essential loss of generality. Agents are endowed with strictly concave, time-separable preferences:

$$u_h(.) = \alpha_h \ln x_h^0 + \beta_h \ln v_h^1\left(x_h^{11}, x_h^{12}\right) + (1 - \alpha_h - \beta_h) \ln v_h^2\left(x_h^{21}, x_h^{22}\right).$$

There are three agents, with

$$\begin{aligned}
v_1^1\left(x_1^{s1}, x_1^{s2}\right) &= \left[(x_1^{s1})^{-2} + k^3 (x_1^{s2})^{-2}\right]^{-\frac{1}{2}}, \text{ for } s = 1, 2, \\
v_2^1\left(x_2^{11}, x_2^{12}\right) &= \left[k^3 (x_2^{11})^{-2} + (x_2^{12})^{-2}\right]^{-\frac{1}{2}}, \quad v_2^2\left(x_2^{21}, x_2^{22}\right) = \left[(x_2^{21})^{-2} + k^3 (x_2^{22})^{-2}\right]^{-\frac{1}{2}}, \\
v_3^1\left(x_3^{11}, x_3^{12}\right) &= \left[(x_3^{11})^{-2} + k^3 (x_3^{12})^{-2}\right]^{-\frac{1}{2}}, \quad v_3^2\left(x_3^{21}, x_3^{22}\right) = \left[k^3 (x_3^{21})^{-2} + (x_3^{22})^{-2}\right]^{-\frac{1}{2}}.
\end{aligned}$$

Endowments are $\omega_1 = (14, (2, 0), (2, 0))$, $\omega_2 = (0, (4, 20), (14, 0))$, and $\omega_3 = (0, (14, 0), (4, 20))$. Set $p^0 = p^{11} = p^{21} = 1$. Given any $\bar{b}_1$ and each vector p, agent 1's associate spot s indirect utility functions are

$$V_1^s\left(p^s, \bar{b}_1\right) \equiv \left(2 + \bar{b}_1\right) \left[\left(\frac{p^{s2\frac{1}{3}}}{p^{s2\frac{1}{3}} + kp^{s2}}\right)^{-2} + k^3 \left(\frac{k}{p^{s2\frac{1}{3}} + kp^{s2}}\right)^{-2}\right]^{-\frac{1}{2}} \equiv \left(2 + \bar{b}_1\right) g_1^s(p^s).$$

The results for agents 2 and 3 are similar. A key property follows from our selection of the utility functions: spot commodity prices affect the choice of the optimal portfolio only because they may determine the value of the spot endowments.[8]

Fix $\bar{b} \equiv \left(\bar{b}_1, \bar{b}_2, \bar{b}_3\right) = (8, -4, -4)$. Given $\bar{b}$, at each spot, the equilibrium is obtained solving the market clearing condition for good 1, i.e.,

$$\frac{10p^{s2\frac{1}{3}}}{kp^{s2} + p^{s2\frac{1}{3}}} + \frac{10p^{s2\frac{1}{3}}}{kp^{s2} + p^{s2\frac{1}{3}}} + \frac{(20p^{s2})kp^{s2\frac{1}{3}}}{p^{s2} + kp^{s2\frac{1}{3}}} - 20 = 0.$$

A convenient feature of this spot economy is that, for the given ω and $\bar{b}$, $\bar{p}^{s2} = 1$, for each s, is an equilibrium for each $k > 0$. For $k \geq \frac{1}{3}$, it is the only equilibrium; for $k < \frac{1}{3}$, there are three equilibria, see Figure 1.[9]

On the other hand, at the equilibrium $\bar{p}^{s2} = 1$, the derivative of the excess demand function

[8] Hence, and due to the log utility functions, the functions $g_h^s(p^s)$ are irrelevant for the optimal portfolio choice.

[9] The spot 1 and 2 subeconomies are based on an example of a CES economy with multiple equilibria proposed by D. Blair.

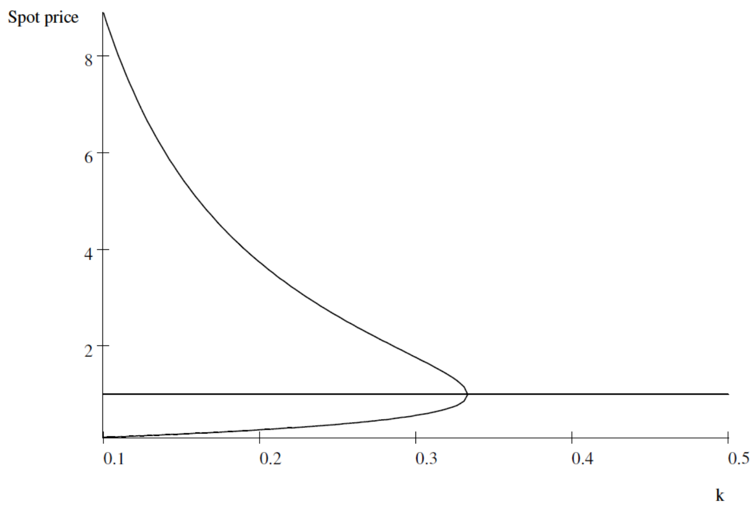

Figure 1:
Equilibrium prices $\sqrt[3]{p^{s2}}$ as a function of k

depends upon k. It is given by

$$\frac{\partial Z^{s1}}{\partial p^{s2}}|_{\overline{p}^{s2}=1} = \left(\frac{20}{3}\frac{3k^2-k}{(k+1)(k+1)}\right),$$

which is obviously equal to zero at $k = \frac{1}{3}$, given the portfolio $\overline{b}$.[10]
Given $\overline{b}$ and $\left(\overline{p}^{12}, \overline{p}^{22}\right) = (1, 1)$, the excess demands for good 2 at spot 1 and 2 are

$$\overline{z}_1^{s2} = \left[\frac{10k}{1+k}, \frac{10k}{1+k}\right], \quad \overline{z}_2^{s2} = \left[-\frac{20k}{1+k}, \frac{10k}{1+k}\right], \quad \overline{z}_3^{s2} = \left[\frac{10k}{1+k}, -\frac{20k}{1+k}\right].$$

Let t be the endowment reallocation, with $t_1 = t$ and $t_2 = t_3 = -\frac{t}{2}$. Consider now the portfolio optimization problems of the three agents. Using the previous observation, we can write them as

$$\begin{aligned}
\max_{b_1} V_1(.) &= \alpha_1 \ln(14 - qb_1 + t) + \beta_1 \ln(2 + b_1) + (1 - \alpha_1 - \beta_1)\ln(2 + b_1) + G_1(p^{12}, p^{22}), \\
\max_{b_2} V_2(.) &= \alpha_2 \ln\left(-qb_2 - \frac{t}{2}\right) + \beta_2 \ln\left(20p^{12} + 4 + b_2\right) + (1 - \alpha_2 - \beta_2)\ln(14 + b_2) + G_2(p^{12}, p^{22}), \\
\max_{b_3} V_3(.) &= \alpha_3 \ln\left(-qb_3 - \frac{t}{2}\right) + \beta_3 \ln(14 + b_3) + (1 - \alpha_3 - \beta_3)\ln\left(20p^{22} + 4 + b_3\right) + G_3(p^{12}, p^{22}).
\end{aligned}$$

It is easy to check that, at $\left(\overline{p}^{12}, \overline{p}^{22}, \overline{q}\right) = (1,1,1)$ and $\overline{t} = 0$, $\left(\overline{b}_1, \overline{b}_2, \overline{b}_3\right) = (8, -4, -4)$ are the optimal portfolios if and only if

$$\alpha_1 = \frac{3}{8}, \ \beta_2 = (2 - 7\alpha_2), \text{ and } \beta_3 = (6\alpha_3 - 1).$$

This, and the positivity constraints on the parameters of the utility functions, imply that both α_2 and α_3 must then be in the interval $\left(\frac{1}{6}, \frac{2}{7}\right)$.
Using these properties and applying the implicit function thm. to the FOCs of the three

[10] The critical equilibrium of the spot economy at $k = \frac{1}{3}$ may, or may not, be associated with a critical equilibrium of the financial economy.

optimization problems, we obtain

$$\begin{aligned}
\frac{\partial b_1}{\partial t} &= \left(\frac{5}{8}\right), \ \frac{\partial b_1}{\partial q} = \left(-\frac{70}{8}\right), \text{ and } \frac{\partial b_1}{\partial p^{12}} = \frac{\partial b_1}{\partial p^{21}} = 0, \\
\frac{\partial b_2}{\partial t} &= \left(\frac{25\alpha_2}{4 - 84\alpha_2}\right), \ \frac{\partial b_2}{\partial p^{12}} = \left(\frac{280\alpha_2 - 80}{84\alpha_2 - 4}\right), \text{ and } \frac{\partial b_2}{\partial p^{22}} = \frac{\partial b_2}{\partial q} = 0, \\
\frac{\partial b_3}{\partial t} &= \left(\frac{25\alpha_3}{4 - 84\alpha_3}\right), \ \frac{\partial b_3}{\partial p^{22}} = \left(\frac{280\alpha_3 - 80}{84\alpha_3 - 4}\right), \text{ and } \frac{\partial b_3}{\partial p^{12}} = \frac{\partial b_3}{\partial q} = 0.
\end{aligned}$$

We can now compute the values of $\left[\frac{\partial q}{\partial t}, \frac{\partial p^{12}}{\partial t}, \frac{\partial p^{22}}{\partial t}\right]$. Define the equilibrium map

$$\Phi(q,p) \equiv \left[\ \textstyle\sum_h b_h(.), \sum_h z_h^{11}(.), \sum_h z_h^{21}(.)\ \right] = 0.$$

By the implicit function theorem,

$$\left[\frac{\partial q}{\partial t}, \frac{\partial p^{12}}{\partial t}, \frac{\partial p^{22}}{\partial t}\right]^T = -\left[D_{(q,p)}\Phi(q,p)\right]^{-1} D_t\Phi(q,p).$$

As shown in Appendix, because of the particular - trichotomous - structure of our economy,

$$\left[\frac{\partial q}{\partial t}, \frac{\partial p^{12}}{\partial t}, \frac{\partial p^{22}}{\partial t},\right]^T = -\left[\frac{\frac{\partial b_1}{\partial t}}{\frac{\partial b_1}{\partial q}}, \ \frac{\frac{\partial b_2}{\partial t}}{\frac{\partial b_2}{\partial p^{12}}}, \ \frac{\frac{\partial b_3}{\partial t}}{\frac{\partial b_3}{\partial p^{22}}}\right].$$

Using the previous results, this implies

$$\left[\frac{\partial q}{\partial t}, \frac{\partial p^{12}}{\partial t}, \frac{\partial p^{22}}{\partial t},\right]^T = \left[\frac{1}{14}, \ \frac{5\alpha_2}{56\alpha_2 - 16}, \ \frac{5\alpha_3}{56\alpha_3 - 16}\right].$$

Consider now the effect of a period 0 endowment reallocation on the equilibrium utilities:

$$\begin{aligned}
\frac{\partial V_1}{\partial t} &= \lambda_1^0 + \lambda_1^0\left(-\bar{b}_1\right)\frac{\partial q}{\partial t} + \lambda_1^1\left(-\bar{z}_1^{12}\right)\frac{\partial p^{12}}{\partial t} + \lambda_1^2\left(-\bar{z}_1^{22}\right)\frac{\partial p^{22}}{\partial t}, \\
\frac{\partial V_h}{\partial t} &= -\frac{\lambda_h^0}{2} + \lambda_h^0\left(-\bar{b}_h\right)\frac{\partial q}{\partial t} + \lambda_h^1\left(-\bar{z}_h^{12}\right)\frac{\partial p^{12}}{\partial t} + \lambda_h^2\left(-\bar{z}_h^{22}\right)\frac{\partial p^{22}}{\partial t}, \text{ for } h > 1.
\end{aligned}$$

Using $\lambda_1 = \left(\frac{3}{48}, \frac{\beta_1}{10}, \frac{\frac{5}{8}-\beta_1}{10}\right)$, $\lambda_2 = \left(\frac{\alpha_2}{4}, \frac{\beta_2}{20}, \frac{1-\alpha_2-\beta_2}{10}\right)$ and $\lambda_3 = \left(\frac{\alpha_3}{4}, \frac{\beta_3}{10}, \frac{1-\alpha_3-\beta_3}{20}\right)$, and the values of the excess demands computed above, we obtain

$$\begin{aligned}
\frac{\partial V_1}{\partial t} &= \frac{3}{112} - \frac{\beta_1}{4}\left(\frac{5\alpha_2}{56\alpha_2 - 16}\right) - \frac{\frac{5}{8} - \beta_1}{4}\left(\frac{5\alpha_3}{56\alpha_3 - 16}\right), \\
\frac{\partial V_2}{\partial t} &= -\frac{3}{56}\alpha_2 - \frac{5}{32}\alpha_2 - \frac{6\alpha_2 - 1}{4}\left(\frac{5\alpha_3}{56\alpha_3 - 16}\right), \\
\frac{\partial V_3}{\partial t} &= -\frac{3}{56}\alpha_3 - \frac{5}{32}\alpha_3 - \frac{6\alpha_3 - 1}{4}\left(\frac{5\alpha_2}{56\alpha_2 - 16}\right).
\end{aligned}$$

For $h = 2, 3$, $\alpha_h < \frac{2}{7}$, and $(56\alpha_h - 16) < 0$. Hence, $\frac{\partial V_1}{\partial t}$ is strictly positive and monotonically increasing in (α_1, α_2). Similarly, $\frac{\partial V_2}{\partial t}$ is monotonically increasing in α_3 and divergent for $\alpha_3 \to \frac{2}{7}$ (while $\frac{\partial V_3}{\partial t}$ has the same properties with respect to α_2). It follows that, for α_2 and α_3 close to $\frac{2}{7}$, $\left(\frac{\partial V_1}{\partial t}, \frac{\partial V_2}{\partial t}, \frac{\partial V_3}{\partial t}\right) >> 0$. This is shown in Figure 2, which reports the values of $\frac{\partial V_1}{\partial t}$ (the thick curve) and $\frac{\partial V_2}{\partial t} = \frac{\partial V_3}{\partial t}$ (the thin one) for a range of values of $\alpha_2 = \alpha_3 \in \left(\frac{1}{6}, \frac{2}{7}\right)$. Evidently, for $\alpha_2 = \alpha_3$ sufficiently large, each equilibrium is not $\omega - CPO$. ■

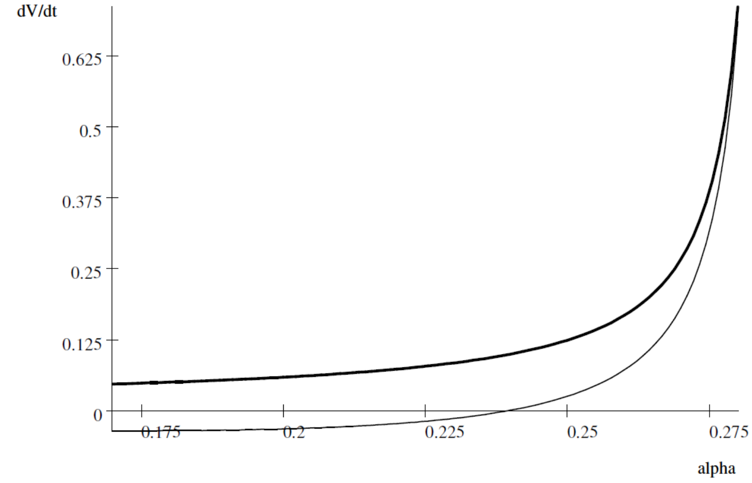

Figure 2:
Values of $\frac{\partial V_1}{\partial t}$, $\frac{\partial V_2}{\partial t}$ and $\frac{\partial V_3}{\partial t}$ as a function of α

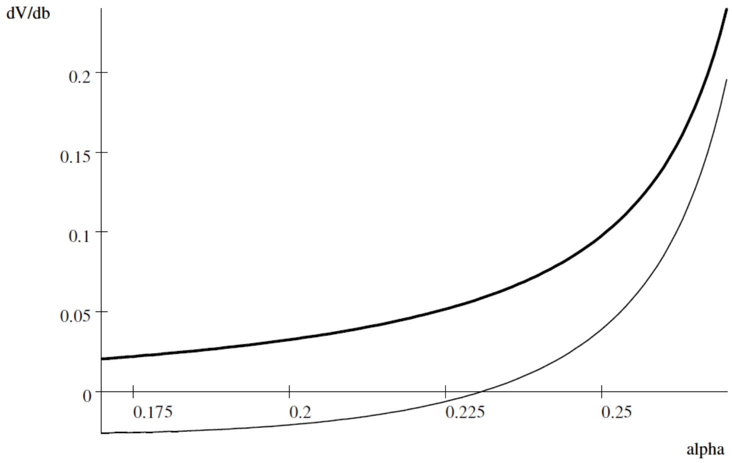

Figure 3:
$\frac{\partial V_h}{\partial b}$ due to the changes in good prices

In the example, we have chosen k so that the spot equilibria are critical. This simplifies a lot the computations, but nothing of relevance rests on it. In Appendix, we propose a different example, with two agents and where spot equilibria are not critical, obtaining a similar result.

Remark 6. The same example can also be used to show that a pure portfolio reallocation of a single asset can be sufficient to guarantee a Pareto improvement: Figure 3 presents the values of the derivatives of the indirect utility functions of the three agents for an arbitrarily given change the portfolios which happens to be identical to the one induced by the endowment reallocation. We just consider the effects of the changes in spot commodity prices in the future periods. There is an open range of values of the parameters such that the utility of each agent is increasing. Once again, the dimension of the policy profile is (H-1), but we can Pareto improve upon the equilibrium allocation.

We can now state our second result in a somewhat more general form, showing that some of the peculiar features of the examples are not essential: they just allow for computational feasibility.

Proposition 7. There are open set of economies (ω, u, R) with equilibria which are not $\omega-$CPO equilibria. This holds for time-separable utility functions, and also for VNM utility functions.

Proof. Example 5 above shows that there are economies with time-separable preferences such that there is an endowment reallocation which Pareto improves upon the original equilibrium. Since the equilibrium is regular, small changes of the parameters will not break down the result, so that it holds for an open set of economies.
The easiest way to establish that a similar result holds even for VNM utility functions is to consider a three period economy, with a realization of uncertainty only at period 2. Consider an economy as the one described in Example A1 in Appendix, but having two states at period 2. Endowments and preferences at the two states of the world, each having probability $\frac{1}{2}$, are identical. It is easy to check that this new (sunspot like) economy has the same market clearing conditions as the economy with three periods and no uncertainty described in Example A1. It is also easy to check that exactly the same computations reported in Appendix allow us to conclude that, for this VNM economy, there is a Pareto improving endowment reallocation. To conclude, perturb period 2 endowments in different directions in the two states, introducing intrinsic uncertainty. Regularity of the equilibrium guarantees that, provided that the perturbations are sufficiently small, $\omega-$CP suboptimality is preserved. ∎

4. Conclusions

We have considered the canonical GEI model with numeraire assets. We have shown that there are open sets of economies such that their equilibria can be improved upon by an appropriate reallocation of period zero initial endowments. We have also shown that, for each economy defined in terms of utility functions and asset payoffs, there are open sets of endowments such that it is impossible to attain any Pareto improvement by pure period zero endowment reallocation. Our result is weaker than the generic one obtainable when the policy profile is the portfolio of each agent, as in Geanakoplos and Polemarchakis (1996) and Citanna et al. (1998). Still, we believe that it settles an open issue in the literature on constrained inefficiency in GEI and that it contributes to a better understanding of this phenomenon.

It remains an open issue under which general conditions existence of an open set of endowments with $\omega-$CP suboptimal equilibria generically holds in the space of the economies defined by asset structure and preferences. Our analysis shows that the key ingredients are the matrix Λ with typical element $[-\lambda_h^s z_h^{sc}]$ (or $\left[-\lambda_h^0 b_h^j\right]$), and the matrices $D_{(p,q)}\Phi(.)$ and $D_t\Phi(.)$. Generically in $(u,\omega)-$space, Λ has full row rank H. This essentially requires some degree of heterogeneity across agents. Next, we need that there is some vector t such that $\left[\Lambda_h \left[D_{(p,q)}\Phi(.)\right]^{-1} D_t\Phi(.)t\right] t_h > 0$ for each h such that $t_h < 0$, i.e., the second order effect must increase the utility of the agents with $t_h < 0$. Finally, we need that these second order effects are sufficiently strong so that they can overcome the (possibly) negative first order effect. This is guaranteed if we are sufficiently close to a critical equilibrium. This motivates our conjecture: provided that an economy (defined by utilities and asset structure), has a critical equilibrium, then, with sufficient heterogeneity, there is some open set of endowments such that at least one equilibrium is not $\omega - CPO$.[11]

5. Appendix

Proof of Lemma 3. Consider any commodity $\overline{sc}$. Then,

[11] This conjecture should remind of Safra (1981), concerning the transfer paradox.

$$
\begin{aligned}
\frac{\partial V_h}{\partial p^{\overline{sc}}} &= \sum_{sc} \frac{\partial u_h}{\partial x_h^{sc}} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} = \sum_s \lambda_h^s \sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} \\
&= -\lambda_h^{\overline{s}} z_h^{\overline{sc}} + \sum_j \left[-\lambda_h^0 q^j \frac{\partial b_h^j}{\partial p^{\overline{sc}}} + \sum_s \lambda_h^s r^{sj} \frac{\partial b_h^j}{\partial p^{\overline{sc}}} \right] = -\lambda_h^{\overline{s}} z_h^{\overline{sc}}
\end{aligned}
$$

The last two equalities are obtained taking the derivative of the budget constraint in spot s,

$$
\begin{aligned}
\sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} &= \sum_j r^{sj} \frac{\partial b_h^j}{\partial p^{\overline{sc}}}, \text{ if } s \neq \overline{s}, \\
\sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial p^{\overline{sc}}} &= -z_h^{\overline{sc}} + r^s \frac{\partial b_h^{sc}}{\partial p^{\overline{sc}}}, \text{ if } s = \overline{s},
\end{aligned}
$$

and taking into account the no arbitrage conditions.

Similarly,

$$
\begin{aligned}
\frac{\partial V_h}{\partial q^{\overline{j}}} &= \sum_{sc} \frac{\partial u_h}{\partial x_h^{sc}} \frac{\partial x_h^{sc}}{\partial q^{\overline{j}}} = \sum_s \lambda_h^s \sum_c p^{sc} \frac{\partial x_h^{sc}}{\partial q^{\overline{j}}} = -\lambda_h^0 b_h^{\overline{j}} + \sum_j \left[-\lambda_h^0 q^j \frac{\partial b_h^j}{\partial q^{\overline{j}}} + \sum_s \lambda_h^s r^{sj} \frac{\partial b_h^j}{\partial q^{\overline{j}}} \right] \\
&= -\lambda_h^0 b_h^{\overline{j}}. \qquad \blacksquare
\end{aligned}
$$

Example 5: Computation of $\left[\frac{\partial q}{\partial t}, \frac{\partial p^{12}}{\partial t}, \frac{\partial p^{22}}{\partial t}\right]$.

Because of the endowment profile, $\frac{\partial FOC_1}{\partial p^{12}} = \frac{\partial FOC_1}{\partial p^{22}} = 0$, $\frac{\partial FOC_2}{\partial q} = \frac{\partial FOC_2}{\partial p^{22}} = 0$ and $\frac{\partial FOC_3}{\partial q} = \frac{\partial FOC_3}{\partial p^{12}} = 0$. Hence,

$$
D_{(q,p^{12},p^{22},t)}\Phi(p,q) = \begin{bmatrix}
\frac{\partial b_1}{\partial q} & \frac{\partial b_2}{\partial p^{12}} & \frac{\partial b_3}{\partial p^{22}} & \frac{\partial b_B}{\partial t} + \frac{\partial b_A}{\partial t} \\
\frac{\partial x_1^{11}}{\partial b_1}\frac{\partial b_1}{\partial q} & \frac{\partial x_2^{11}}{\partial b_2}\frac{\partial b_2}{\partial p^{12}} + \frac{\partial \overline{Z}^{11}}{\partial p^{12}} & 0 & \sum_h \frac{\partial x_h^{11}}{\partial b_h}\frac{\partial b_h}{\partial t} \\
\frac{\partial x_1^{21}}{\partial b_1}\frac{\partial b_1}{\partial q} & 0 & \frac{\partial x_3^{12}}{\partial b_3}\frac{\partial b_3}{\partial p^{22}} + \frac{\partial \overline{Z}^{21}}{\partial p^{22}} & \sum_h \frac{\partial x_h^{21}}{\partial b_h}\frac{\partial b_h}{\partial t}
\end{bmatrix},
$$

where $\frac{\partial \overline{Z}^{s1}}{\partial p^{s2}}$ is computed for a given portfolio $\overline{b}$, $\frac{\partial \overline{Z}^{s1}}{\partial p^{s2}}|_{\overline{p}=1} = \left[\frac{20}{3} \; \frac{3k^2-k}{(k+1)(k+1)}\right]$.

Evidently,

$$
\begin{aligned}
\frac{\partial x_1^{s1}}{\partial b_1}|_{\overline{p}^{s2}=1} &= \frac{1}{k+1}, \left[\frac{\partial x_2^{11}}{\partial b_2}|_{\overline{p}^{12}=1}, \frac{\partial x_2^{21}}{\partial b_2}|_{\overline{p}^{22}=1}\right] = \left[\frac{k}{k+1}, \frac{1}{1+k}\right], \text{ and} \\
\left[\frac{\partial x_3^{11}}{\partial b_3}|_{\overline{p}^{12}=1}, \frac{\partial x_3^{21}}{\partial b_3}|_{\overline{p}^{22}=1}\right] &= \left[\frac{1}{k+1}, \frac{k}{1+k}\right]
\end{aligned}
$$

Set $k = \frac{1}{3}$, so that $\frac{\partial \overline{Z}^{s1}}{\partial p^{s2}} = 0$. Then, we can write

$$
D_{(q,p)}\Phi(p,q) = \begin{bmatrix}
1 & 1 & 1 \\
\frac{1}{k+1} & \frac{k}{1+k} & \frac{1}{k+1} \\
\frac{1}{k+1} & \frac{1}{k+1} & \frac{k}{1+k}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial b_1}{\partial q} & 0 & 0 \\
0 & \frac{\partial b_2}{\partial p^{12}} & 0 \\
0 & 0 & \frac{\partial b_3}{\partial p^{22}}
\end{bmatrix},
$$

and

$$\left[D_{(q,p)}\Phi(p,q)\right]^{-1} = \frac{(k+1)^2}{(k-1)^2}\begin{bmatrix} 1/\frac{\partial b_1}{\partial q} & 0 & 0 \\ 0 & 1/\frac{\partial b_2}{\partial p^{12}} & 0 \\ 0 & 0 & 1/\frac{\partial b_3}{\partial p^{22}} \end{bmatrix}\begin{bmatrix} \frac{k-1}{k+1} & -\frac{k-1}{k+1} & \frac{(-k+1)}{k+1} \\ -\frac{k-1}{(k+1)^2} & \frac{k-1}{k+1} & 0 \\ \frac{-k+1}{(k+1)^2} & 0 & \frac{k-1}{k+1} \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \frac{\partial q}{\partial t} \\ \frac{\partial p^{12}}{\partial t} \\ \frac{\partial p^{22}}{\partial t} \end{bmatrix} = -\frac{(k+1)^2}{(k-1)^2}\begin{bmatrix} 1/\frac{\partial b_1}{\partial q} & 0 & 0 \\ 0 & 1/\frac{\partial b_2}{\partial p^{12}} & 0 \\ 0 & 0 & 1/\frac{\partial b_3}{\partial p^{22}} \end{bmatrix}\begin{bmatrix} \frac{(k-1)^2}{(k+1)^2} & 0 & 0 \\ 0 & \frac{(k-1)^2}{(k+1)^2} & 0 \\ 0 & 0 & \frac{(k-1)^2}{(k+1)^2} \end{bmatrix}\begin{bmatrix} \frac{\partial b_1}{\partial t} \\ \frac{\partial b_2}{\partial t} \\ \frac{\partial b_3}{\partial t} \end{bmatrix}$$

$$= \left[-\frac{\frac{\partial b_1}{\partial t}}{\frac{\partial b_1}{\partial q}}, -\frac{\frac{\partial b_2}{\partial t}}{\frac{\partial b_2}{\partial p^{12}}}, -\frac{\frac{\partial b_3}{\partial t}}{\frac{\partial b_3}{\partial p^{22}}}\right]^T.$$

Example A1: There is just one commodity at time 0 and at time 2. This entails no essential loss of generality. Preferences of agent 1 and 2 at spot 1 are as in the previous example. The utility functions are

$$u_h(.) = \alpha_h \ln x_h^0 + \beta_h \ln v_h^1\left(x_h^{11}, x_h^{12}\right) + (1-\alpha_h-\beta_h)\ln x_h^2.$$

Endowments are $\omega_1 = (10,(6,0),0)$, and $\omega_2 = (0,(4,10),10)$.
Essentially as above, and omitting the redundant superscript for the price of commodity 2 at spot 1,

$$x_1^1 = \left(\frac{(6+\overline{b}_1)\,p^{\frac{1}{3}}}{kp+p^{\frac{1}{3}}}, \frac{(6+\overline{b}_1)\,k}{kp+p^{\frac{1}{3}}}\right), \qquad x_2^1 = \left(\frac{(10p+4+\overline{b}_2)\,kp^{\frac{1}{3}}}{kp^{\frac{1}{3}}+p}, \frac{10p+4+\overline{b}_2}{kp^{\frac{1}{3}}+p}\right).$$

and

$$V_1^1(p,\overline{b}_1) \equiv (6+\overline{b}_1)\left[\left(\frac{p^{\frac{1}{3}}}{kp+p^{\frac{1}{3}}}\right)^{-2} + k^3\left(\frac{kp^{-\frac{1}{3}}}{kp+p^{\frac{1}{3}}}\right)^{-2}\right]^{-\frac{1}{2}} \equiv (6+\overline{b}_1)\,g_1^1(p).$$

The result for agents 2 is similar. For the given ω and $\overline{b}$, $\overline{p}=1$ is an equilibrium for each $k>0$. The derivative of the excess demand function depends upon k. It is given by $\frac{\partial \overline{Z}^{11}}{\partial p}|_{\overline{p}=1} = \left(\frac{10}{3}\frac{3k^2-k}{(k+1)(k+1)}\right)$.

Evidently,

$$D_{(q,p,t)}\Phi(p,q) = \begin{bmatrix} \frac{\partial b_1}{\partial q} & \frac{\partial b_2}{\partial p} & \sum_h \frac{\partial b_h}{\partial t} \\ \frac{\partial x_1^{11}}{\partial b_1}\frac{\partial b_1}{\partial q} & \frac{\partial x_2^{11}}{\partial b_2}\frac{\partial b_2}{\partial p} + \frac{\partial \overline{Z}^{11}}{\partial p^{12}} & \sum_h \frac{\partial x_h^{11}}{\partial b_h}\frac{\partial b_h}{\partial t} \end{bmatrix},$$

and we can rewrite the two blocks as

$$
\begin{aligned}
D_{(q,p)}\Phi(p,q) &= \begin{bmatrix} 1 & 1 \\ \frac{\partial x_1^{11}}{\partial b_1} & \frac{\partial x_2^{11}}{\partial b_2} + \frac{\frac{\partial \overline{z}^{11}}{\partial p^{12}}}{\frac{\partial b_2}{\partial p}} \end{bmatrix} \begin{bmatrix} \frac{\partial b_1}{\partial q} & 0 \\ 0 & \frac{\partial b_2}{\partial p} \end{bmatrix} \\
D_{(t)}\Phi(p,q) &= \begin{bmatrix} 1 & 1 \\ \frac{\partial x_1^{11}}{\partial b_1} & \frac{\partial x_2^{11}}{\partial b_2} \end{bmatrix} \begin{bmatrix} \frac{\partial b_1}{\partial t} \\ \frac{\partial b_2}{\partial t} \end{bmatrix}
\end{aligned}
$$

Let $\det \equiv \left[\frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} - \frac{\partial x_1}{\partial b_1} + \frac{\partial x_2}{\partial b_2}\right] = \left[\frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} + \frac{k-1}{1+k}\right]$. Then,

$$
\begin{bmatrix} 1 & 1 \\ \frac{1}{1+k} & \frac{k}{1+k} + \frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} \end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix} \frac{k}{1+k} + \frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} & -1 \\ -\frac{1}{1+k} & 1 \end{bmatrix},
$$

and

$$
\left[\frac{\partial q}{\partial t}, \frac{\partial p}{\partial t}\right]^T = -D_{(q,p)}\Phi(q,p)^{-1} D_t\Phi(q,p) \tag{1}
$$

$$
\begin{aligned}
&= \frac{-1}{\det}\begin{bmatrix} \frac{1}{\frac{\partial b_1}{\partial q}} & 0 \\ 0 & \frac{1}{\frac{\partial b_2}{\partial p}} \end{bmatrix} \times \begin{bmatrix} \frac{k}{1+k} + \frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} & -1 \\ -\frac{1}{1+k} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{1+k} & \frac{k}{1+k} \end{bmatrix} \begin{bmatrix} \frac{\partial b_1}{\partial t} \\ \frac{\partial b_2}{\partial t} \end{bmatrix} \\
&= \left[-\frac{\frac{\partial b_1}{\partial t}}{\frac{\partial b_1}{\partial q}} - \frac{\frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_1}{\partial q}}\frac{\frac{\partial b_2}{\partial t}}{\frac{\partial b_2}{\partial p}}}{\left[\frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} + \frac{k-1}{1+k}\right]}, \quad \frac{-\frac{k-1}{1+k}\frac{\frac{\partial b_2}{\partial t}}{\frac{\partial b_2}{\partial p}}}{\left[\frac{\frac{\partial \widehat{z}^{11}}{\partial p}|_{\overline{p}=1}}{\frac{\partial b_2}{\partial p}} + \frac{k-1}{1+k}\right]}\right]^T
\end{aligned}
$$

To conclude, we need to compute $\left(\frac{\partial b_1}{\partial q}, \frac{\partial b_1}{\partial t}\right)$ and $\left(\frac{\partial b_2}{\partial p}, \frac{\partial b_2}{\partial t}\right)$ using the implicit function thm. applied to the first order conditions of the portfolio optimization problem. First, observe that optimality of $(\overline{b}_1, \overline{b}_2) = (4, -4)$ at $(\overline{q}, \overline{p}) = (1, 1)$ and the nonnegativity constraint on the values of $(\alpha_h, \beta_h, 1-\alpha_h-\beta_h)$ require that $\beta_1(\alpha_1) = \left(\frac{5}{3} - \frac{25}{9}\alpha_1\right)$ and $\alpha_1 \in \left(\frac{3}{8}, \frac{3}{5}\right)$, while $\beta_2(\alpha_2) = \left(\frac{5}{2} - \frac{25}{4}\alpha_2\right)$ and $\alpha_2 \in \left(\frac{2}{7}, \frac{2}{5}\right)$.

Consider a negative transfer for agent 1. By direct computation, at $t = 0$, $\overline{p} = \overline{q} = 1$, $\overline{b}_1 = -\overline{b}_2 = 4$, and using $\beta(\alpha), \delta(\gamma)$:

$$
\begin{aligned}
\frac{\partial FOC_1}{\partial t} &= -\frac{20}{3}\alpha_1, \quad \frac{\partial FOC_1}{\partial q} = -\frac{200}{3}\alpha_1, \quad \frac{\partial FOC_1}{\partial q} = 0, \quad \frac{\partial FOC_1}{\partial b_1} = 6 - \frac{80}{3}\alpha_1, \\
\frac{\partial FOC_2}{\partial t} &= 15\alpha_2, \quad \frac{\partial FOC_2}{\partial q} = 0, \quad \frac{\partial FOC_2}{\partial p} = (150\alpha_2 - 60), \quad \frac{\partial FOC_2}{\partial b_2} = (4 - 35\alpha_2),
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial b_1}{\partial t} &= \frac{20\alpha_1}{18 - 80\alpha_1}, \quad \frac{\partial b_1}{\partial q} = \frac{200\alpha_1}{18 - 80\alpha_1}, \quad \frac{\partial b_1}{\partial p} = 0 \\
&\text{and} \\
\frac{\partial b_2}{\partial t} &= \frac{-15\alpha_2}{4 - 35\alpha_2}, \quad \frac{\partial b_2}{\partial q} = 0, \quad \frac{\partial b_2}{\partial p} = -\frac{150\alpha_2 - 60}{4 - 35\alpha_2}.
\end{aligned}
$$

Since $\lambda_1 = \left(\frac{\alpha_1}{6}, \frac{\beta_1}{10}, \frac{1-\alpha_1-\beta_1}{4}\right)$ and $\lambda_2 = \left(\frac{\alpha_2}{4}, \frac{\beta_2}{10}, \frac{1-\alpha_2-\beta_2}{6}\right)$, while $\overline{z}_1^{12} = \left(\frac{10k}{1+k}\right)$, and $\overline{z}_2^{12} =$

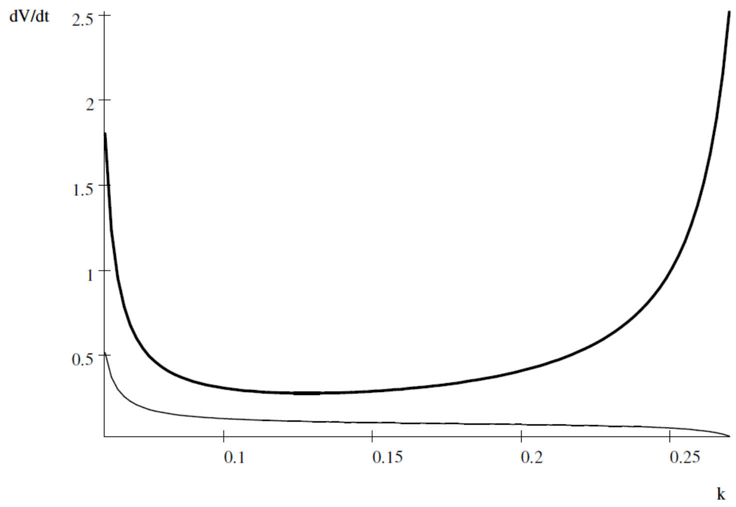

Figure 4:
$\frac{\partial V_1}{\partial t}$ and $\frac{\partial V_2}{\partial t}$ at different values of k

$\left(-\frac{10k}{1+k}\right)$, replacing (β_1, β_2) with $(\beta_1(\alpha_1), \beta_2(\alpha_2))$, we obtain

$$\begin{aligned} \frac{\partial V_1}{\partial t} &= \left(-1 - 4\frac{\partial q}{\partial t}\right)\frac{\alpha_1}{6} - \left(\frac{5}{3} - \frac{25}{9}\alpha_1\right)\left(\frac{k}{1+k}\right)\frac{\partial p}{\partial t} \\ \frac{\partial V_2}{\partial t} &= \left(1 + 4\frac{\partial q}{\partial t}\right)\frac{\alpha_2}{4} + \left(\frac{5}{2} - \frac{25}{4}\alpha_2\right)\left(\frac{k}{1+k}\right)\frac{\partial p}{\partial t}. \end{aligned} \tag{2}$$

Set $\alpha_1 = \frac{38}{100}$ and $\alpha_2 = \frac{39}{100}$. Then,

$$\begin{aligned} \frac{\partial q}{\partial t} &= \frac{1}{190}\left[\frac{40k + 51k^2 - 171}{570k^2 - 193k + 9}\right], \\ \frac{\partial p}{\partial t} &= -\frac{351}{10}\left[\frac{(k+1)(k-1)}{570k^2 - 193k + 9}\right], \end{aligned}$$

and $(\overline{q}, \overline{p}) = (1, 1)$ is a critical equilibrium for $k = \left(\frac{193}{1140} \pm \frac{1}{1140}\sqrt{16\,729}\right)$. The rates of change of the indirect utilities are

$$\begin{aligned} \frac{\partial V_1}{\partial t} &= \left(-1 - \frac{4}{190}\left(\frac{40k + 51k^2 - 171}{570k^2 - 193k + 9}\right)\right)\frac{38}{600} + \frac{3861}{180}\left(\frac{k}{1+k}\right)\left(\frac{(k+1)(k-1)}{570k^2 - 193k + 9}\right) \\ \frac{\partial V_2}{\partial t} &= \left(1 + \frac{4}{190}\left(\frac{40k + 51k^2 - 171}{570k^2 - 193k + 9}\right)\right)\frac{39}{400} - \frac{351}{160}\left(\frac{k}{1+k}\right)\left(\frac{(k+1)(k-1)}{570k^2 - 193k + 9}\right) \end{aligned}$$

Figure 4 shows their values[12] for $k \in \left(\frac{7}{100}, \frac{27}{100}\right)$, an interval contained in one of the connected components of the equilibrium manifold, defined with respect to k. For values of k in this range, the equilibrium with $(\overline{p}, \overline{q}) = (1, 1)$ is clearly not $\omega-$CPO, since it can be improved upon by a small reallocation of period 0 endowment with $t < 0$.

References

[1] Citanna, A., Kajii, A., Villanacci, A., 1998, Constrained Suboptimality in Incomplete Markets: A General Approach and Two Applications, Econ. Theory 11, 495-521.

[12] The solid line describes the values of $\frac{\partial V_A}{\partial t}$.

[2] Citanna, A., Polemarchakis, H.M., Tirelli, M., 2006, The taxation of trades in assets, Journal of Economic Theory 126, 299 – 313.

[3] Geanakoplos, J., and H.M. Polemarchakis, 1986, Existence, Regularity and Constrained Suboptimality of Competitive Allocations When the Asset Market Is Incomplete, in W.P. Heller, R.M. Starr and D. Starrett (eds.), Uncertainty, Information and Communication: Essays in Honor of K.J. Arrow, Vol. III. Cambridge University Press, Cambridge, 65-95.

[4] Hart, O.J., 1975, On the Optimality of Equilibrium When the Market Structure Is Incomplete, J. Econ. Theory 11, 418-443.

[5] Herings, P.J.-J., and H. Herakles Polemarchakis, 2005, Pareto improving price regulation when the asset market is incomplete, Economic Theory 25, 135–154.

[6] Kajii, A., 1994, Anonymity and optimality of competitive equilibria when markets are incomplete, J. Econ. Theory 64, 115–129.

[7] Pietra, T., 2004, Sunspots, Indeterminacy and Pareto Inefficiency in Economies with Incomplete Asset Markets, Economic Theory 24, 687-699.

[8] Safra, Z., 1984, On the Frequency of the Transfer Paradox, Economics Letters 15, 209-212.

[9] Salto, M., Pietra, T., 2013, Welfare and Excess Volatility of Exchange Rates, Economic Theory 52, 501–529.

[10] Stiglitz, J.E., 1982, The Inefficiency of the Stock Market Equilibrium, R. Econ. Studies 49, 241-261.

[11] Tinbergen, J., 1956, Economic Policy: Problems and Design, North-Holland, Amsterdam.

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IAB-Discussion Paper 1/2016

13 January 2016

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ISSN 2195-2663

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